

Polar Complex Numbers in n Dimensions

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Abstract

Polar commutative n -complex numbers of the form $u = x_0 + h_1x_1 + h_2x_2 + \cdots + h_{n-1}x_{n-1}$ are introduced in n dimensions, the variables x_0, \dots, x_{n-1} being real numbers. The polar n -complex number can be represented, in an even number of dimensions, by the modulus d , by the amplitude ρ , by 2 polar angles θ_+, θ_- , by $n/2 - 2$ planar angles ψ_{k-1} , and by $n/2 - 1$ azimuthal angles ϕ_k . In an odd number of dimensions, the polar n -complex number can be represented by d, ρ , by 1 polar angle θ_+ , by $(n - 3)/2$ planar angles ψ_{k-1} , and by $(n - 1)/2$ azimuthal angles ϕ_k . The exponential function of a polar n -complex number can be expanded in terms of the polar n -dimensional cosexponential functions $g_{nk}(y), k = 0, 1, \dots, n - 1$. Expressions are given for these cosexponential functions. The polar n -complex numbers can be written in exponential and trigonometric forms with the aid of the modulus, amplitude and the angular variables. The polar n -complex functions defined by series of powers are analytic, and the partial derivatives of the components of the polar n -complex functions are closely related. The integrals of polar n -complex functions are independent of path in regions where the functions are regular. The fact that the exponential form of

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a polar n-complex numbers depends on the cyclic variables ϕ_k leads to the concept of pole and residue for integrals on closed paths. The polynomials of polar n-complex variables can be written as products of linear or quadratic factors, although the factorization may not be unique.

1 Introduction

A regular, two-dimensional complex number $x + iy$ can be represented geometrically by the modulus $\rho = (x^2 + y^2)^{1/2}$ and by the polar angle $\theta = \arctan(y/x)$. The modulus ρ is multiplicative and the polar angle θ is additive upon the multiplication of ordinary complex numbers.

The quaternions of Hamilton are a system of hypercomplex numbers defined in four dimensions, the multiplication being a noncommutative operation, [1] and many other hypercomplex systems are possible, [2]-[4] but these hypercomplex systems do not have all the required properties of regular, two-dimensional complex numbers which rendered possible the development of the theory of functions of a complex variable.

A system of complex numbers in n dimensions is described in this work, for which the multiplication is both associative and commutative, and which is rich enough in properties so that an exponential form exists and the concepts of analytic n-complex function, contour integration and residue can be defined. The n-complex numbers introduced in this work have the form $u = x_0 + h_1x_1 + h_2x_2 + \cdots + h_{n-1}x_{n-1}$, the variables x_0, \dots, x_{n-1} being real numbers. The multiplication rules for the complex units h_1, \dots, h_{n-1} are $h_jh_k = h_{j+k}$ if $0 \leq j+k \leq n-1$, and $h_jh_k = h_{j+k-n}$ if $n \leq j+k \leq 2n-2$. The product of two n-complex numbers is equal to zero if both numbers are equal to zero, or if the numbers belong to certain n-dimensional hyperplanes described further in this work.

If the n-complex number $u = x_0 + h_1x_1 + h_2x_2 + \cdots + h_{n-1}x_{n-1}$ is represented by the point A of coordinates x_0, x_1, \dots, x_{n-1} , the position of the point A can be described, in an even number of dimensions, by the modulus $d = (x_0^2 + x_1^2 + \cdots + x_{n-1}^2)^{1/2}$, by $n/2 - 1$ azimuthal angles ϕ_k , by $n/2 - 2$ planar angles ψ_{k-1} , and by 2 polar angles θ_+, θ_- . In an odd number of dimensions, the position of the point A is described by d , by $(n-1)/2$ azimuthal

angles ϕ_k , by $(n-3)/2$ planar angles ψ_{k-1} , and by 1 polar angle θ_+ . An amplitude ρ can be defined for even n as $\rho^n = v_+ v_- \rho_1^2 \cdots \rho_{n/2-1}^2$, and for odd n as $\rho^n = v_+ \rho_1^2 \cdots \rho_{(n-1)/2}^2$, where $v_+ = x_0 + \cdots + x_{n-1}$, $v_- = x_0 - x_1 + \cdots + x_{n-2} - x_{n-1}$, and ρ_k are radii in orthogonal two-dimensional planes defined further in this work. The amplitude ρ , the variables v_+, v_- , the radii ρ_k , the variables $(1/\sqrt{2}) \tan \theta_+, (1/\sqrt{2}) \tan \theta_-, \tan \psi_{k-1}$ are multiplicative, and the azimuthal angles ϕ_k are additive upon the multiplication of n-complex numbers. Because of the role of the axis v_+ and, in an even number of dimensions, of the axis v_- , in the description of the position of the point A with the aid of the polar angle θ_+ and, in an even number of dimensions, of the polar angle θ_- , the hypercomplex numbers studied in this work will be called polar n-complex number, to distinguish them from the planar n-complex numbers, which exist in an even number of dimensions. [5]

The exponential function of an n-complex number can be expanded in terms of the polar n-dimensional cosexponential functions $g_{nk}(y) = \sum_{p=0}^{\infty} y^{k+pn} / (k+pn)!$, $k = 0, 1, \dots, n-1$. It is shown that $g_{nk}(y) = \frac{1}{n} \sum_{l=0}^{n-1} \exp \{y \cos(2\pi l/n)\} \cos \{y \sin(2\pi l/n) - 2\pi kl/n\}$, $k = 0, 1, \dots, n-1$. Addition theorems and other relations are obtained for the polar n-dimensional cosexponential functions.

The exponential form of an n-complex number, which in an even number of dimensions n can be defined for $x_0 + \cdots + x_{n-1} > 0, x_0 - x_1 + \cdots + x_{n-2} - x_{n-1} > 0$, is $u = \rho \exp \left\{ \sum_{p=1}^{n-1} h_p \left[(1/n) \ln \sqrt{2} / \tan \theta_+ + ((-1)^p / n) \ln \sqrt{2} / \tan \theta_- - (2/n) \sum_{k=2}^{n/2-1} \cos(2\pi kp/n) \ln \tan \psi_{k-1} \right] \right\} \exp \left(\sum_{k=1}^{n/2-1} \tilde{e}_k \phi_k \right)$, where $\tilde{e}_k = (2/n) \sum_{p=1}^{n-1} h_p \sin(2\pi pk/n)$. In an odd number of dimensions n , the exponential form exists for $x_0 + \cdots + x_{n-1} > 0$, and is $u = \rho \exp \left\{ \sum_{p=1}^{n-1} h_p \left[(1/n) \ln \sqrt{2} / \tan \theta_+ - (2/n) \sum_{k=2}^{(n-1)/2} \cos(2\pi kp/n) \ln \tan \psi_{k-1} \right] \right\} \exp \left(\sum_{k=1}^{(n-1)/2} \tilde{e}_k \phi_k \right)$. A trigonometric form also exists for an n-complex number u , when u is written as the product of the modulus d , of a factor depending on the polar and planar angles $\theta_+, \theta_-, \psi_{k-1}$ and of an exponential factor depending on the azimuthal angles ϕ_k .

Expressions are given for the elementary functions of n-complex variable. The functions $f(u)$ of n-complex variable which are defined by power series have derivatives independent of the direction of approach to the point under consideration. If the n-complex function $f(u)$ of the n-complex variable u is written in terms of the real functions $P_k(x_0, \dots, x_{n-1})$, $k = 0, \dots, n-1$, then relations of equality exist between partial derivatives of the functions P_k . The

integral $\int_A^B f(u)du$ of an n -complex function between two points A, B is independent of the path connecting A, B , in regions where f is regular. If $f(u)$ is an analytic n -complex function, then $\oint_{\Gamma} f(u)du/(u - u_0) = 2\pi f(u_0) \sum_{k=1}^{[(n-1)/2]} \tilde{e}_k \text{ int}(u_{0\xi_k\eta_k}, \Gamma_{\xi_k\eta_k})$, where the functional int takes the values 0 or 1 depending on the relation between $u_{0\xi_k\eta_k}$ and $\Gamma_{\xi_k\eta_k}$, which are respectively the projections of the point u_0 and of the loop Γ on the plane defined by the orthogonal axes ξ_k and η_k , as explained further in this work.

A polynomial $u^m + a_1 u^{m-1} + \dots + a_{m-1} u + a_m$ can be written as a product of linear or quadratic factors, although the factorization may not be unique.

This paper belongs to a series of studies on commutative complex numbers in n dimensions. [6] A detailed analysis of the cases for $n = 2, 3, 4, 5, 6$ of the polar n -complex numbers can be found in the corresponding studies mentioned in Ref. [6].

2 Operations with polar n -complex numbers

A complex number in n dimensions is determined by its n components $(x_0, x_1, \dots, x_{n-1})$. The polar n -complex numbers and their operations discussed in this work can be represented by writing the n -complex number $(x_0, x_1, \dots, x_{n-1})$ as $u = x_0 + h_1 x_1 + h_2 x_2 + \dots + h_{n-1} x_{n-1}$, where h_1, h_2, \dots, h_{n-1} are bases for which the multiplication rules are

$$h_j h_k = h_l, \quad l = j + k - n[(j + k)/n], \quad (1)$$

for $j, k, l = 0, 1, \dots, n - 1$. In Eq. (1), $[(j + k)/n]$ denotes the integer part of $(j + k)/n$, the integer part being defined as $[a] \leq a < [a] + 1$, so that $0 \leq j + k - n[(j + k)/n] \leq n - 1$. In this work, brackets larger than the regular brackets $[]$ do not have the meaning of integer part. The significance of the composition laws in Eq. (1) can be understood by representing the bases h_j, h_k by points on a circle at the angles $\alpha_j = 2\pi j/n, \alpha_k = 2\pi k/n$, as shown in Fig. 1, and the product $h_j h_k$ by the point of the circle at the angle $2\pi(j + k)/n$. If $2\pi \leq 2\pi(j + k)/n < 4\pi$, the point represents the basis h_l of angle $\alpha_l = 2\pi(j + k - n)/n$.

Two n -complex numbers $u = x_0 + h_1 x_1 + h_2 x_2 + \dots + h_{n-1} x_{n-1}$, $u' = x'_0 + h_1 x'_1 + h_2 x'_2 + \dots + h_{n-1} x'_{n-1}$ are equal if and only if $x_i = x'_i, i = 0, 1, \dots, n - 1$. The sum of the n -complex numbers u and u' is

$$u + u' = x_0 + x'_0 + h_1(x_1 + x'_1) + \dots + h_{n-1}(x_{n-1} + x'_{n-1}). \quad (2)$$

The product of the numbers u, u' is

$$\begin{aligned}
uu' &= x_0x'_0 + x_1x'_{n-1} + x_2x'_{n-2} + x_3x'_{n-3} + \cdots + x_{n-1}x'_1 \\
&+ h_1(x_0x'_1 + x_1x'_0 + x_2x'_{n-1} + x_3x'_{n-2} + \cdots + x_{n-1}x'_2) \\
&+ h_2(x_0x'_2 + x_1x'_1 + x_2x'_0 + x_3x'_{n-1} + \cdots + x_{n-1}x'_3) \\
&\vdots \\
&+ h_{n-1}(x_0x'_{n-1} + x_1x'_{n-2} + x_2x'_{n-3} + x_3x'_{n-4} + \cdots + x_{n-1}x'_0).
\end{aligned} \tag{3}$$

The product uu' can be written as

$$uu' = \sum_{k=0}^{n-1} h_k \sum_{l=0}^{n-1} x_l x'_{k-l+n[(n-k-1+l)/n]}. \tag{4}$$

If u, u', u'' are n -complex numbers, the multiplication is associative

$$(uu')u'' = u(u'u'') \tag{5}$$

and commutative

$$uu' = u'u, \tag{6}$$

because the product of the bases, defined in Eq. (1), is associative and commutative. The fact that the multiplication is commutative can be seen also directly from Eq. (3). The n -complex zero is $0 + h_1 \cdot 0 + \cdots + h_{n-1} \cdot 0$, denoted simply 0, and the n -complex unity is $1 + h_1 \cdot 0 + \cdots + h_{n-1} \cdot 0$, denoted simply 1.

The inverse of the n -complex number $u = x_0 + h_1x_1 + h_2x_2 + \cdots + h_{n-1}x_{n-1}$ is the n -complex number $u' = x'_0 + h_1x'_1 + h_2x'_2 + \cdots + h_{n-1}x'_{n-1}$ having the property that

$$uu' = 1. \tag{7}$$

Written on components, the condition, Eq. (7), is

$$\begin{aligned}
x_0x'_0 + x_1x'_{n-1} + x_2x'_{n-2} + x_3x'_{n-3} + \cdots + x_{n-1}x'_1 &= 1, \\
x_0x'_1 + x_1x'_0 + x_2x'_{n-1} + x_3x'_{n-2} + \cdots + x_{n-1}x'_2 &= 0, \\
x_0x'_2 + x_1x'_1 + x_2x'_0 + x_3x'_{n-1} + \cdots + x_{n-1}x'_3 &= 0, \\
&\vdots \\
x_0x'_{n-1} + x_1x'_{n-2} + x_2x'_{n-3} + x_3x'_{n-4} + \cdots + x_{n-1}x'_0 &= 0.
\end{aligned} \tag{8}$$

The system (8) has a solution provided that the determinant of the system,

$$\nu = \det(A), \tag{9}$$

is not equal to zero, $\nu \neq 0$, where

$$A = \begin{pmatrix} x_0 & x_{n-1} & x_{n-2} & \cdots & x_1 \\ x_1 & x_0 & x_{n-1} & \cdots & x_2 \\ x_2 & x_1 & x_0 & \cdots & x_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{n-1} & x_{n-2} & x_{n-3} & \cdots & x_0 \end{pmatrix}. \quad (10)$$

If $\nu > 0$, the quantity

$$\rho = \nu^{1/n} \quad (11)$$

will be called amplitude of the n-complex number $u = x_0 + h_1x_1 + h_2x_2 + \cdots + h_{n-1}x_{n-1}$.

The quantity ν can be written as a product of linear factors

$$\nu = \prod_{k=0}^{n-1} \left(x_0 + \epsilon_k x_1 + \epsilon_k^2 x_2 + \cdots + \epsilon_k^{n-1} x_{n-1} \right), \quad (12)$$

where $\epsilon_k = e^{2\pi i k/n}$, i being the imaginary unit. The factors appearing in Eq. (12) are of the form

$$x_0 + \epsilon_k x_1 + \epsilon_k^2 x_2 + \cdots + \epsilon_k^{n-1} x_{n-1} = v_k + i\tilde{v}_k, \quad (13)$$

where

$$v_k = \sum_{p=0}^{n-1} x_p \cos \frac{2\pi k p}{n}, \quad (14)$$

$$\tilde{v}_k = \sum_{p=0}^{n-1} x_p \sin \frac{2\pi k p}{n}, \quad (15)$$

for $k = 1, 2, \dots, n-1$ and, if n is even, $k \neq n/2$. For $k = 0$ the factor in Eq. (13) is

$$v_+ = x_0 + x_1 + \cdots + x_{n-1}, \quad (16)$$

and if n is even, for $k = n/2$ the factor in Eq. (13) is

$$v_- = x_0 - x_1 + \cdots + x_{n-2} - x_{n-1}. \quad (17)$$

It can be seen that $v_k = v_{n-k}$, $\tilde{v}_k = -\tilde{v}_{n-k}$, $k = 1, \dots, [(n-1)/2]$. The variables $v_+, v_-, v_k, \tilde{v}_k$, $k = 1, \dots, [(n-1)/2]$ will be called canonical polar n-complex variables. Therefore, the factors appear in Eq. (12) in complex-conjugate pairs of the form $v_k + i\tilde{v}_k$ and $v_{n-k} + i\tilde{v}_{n-k} = v_k - i\tilde{v}_k$,

where $k = 1, \dots, [(n-1)/2]$, so that the product ν is a real quantity. If n is an even number, the quantity ν is

$$\nu = v_+ v_- \prod_{k=1}^{n/2-1} (v_k^2 + \tilde{v}_k^2), \quad (18)$$

and if n is an odd number, ν is

$$\nu = v_+ \prod_{k=0}^{(n-1)/2} (v_k^2 + \tilde{v}_k^2). \quad (19)$$

Thus, in an even number of dimensions n , an n -complex number has an inverse unless it lies on one of the nodal hypersurfaces $x_0 + x_1 + \dots + x_{n-1} = 0$, or $x_0 - x_1 + \dots + x_{n-2} - x_{n-1} = 0$, or $v_1 = 0, \tilde{v}_1 = 0, \dots$, or $v_{n/2-1} = 0, \tilde{v}_{n/2-1} = 0$. In an odd number of dimensions n , an n -complex number has an inverse unless it lies on one of the nodal hypersurfaces $x_0 + x_1 + \dots + x_{n-1} = 0$, or $v_1 = 0, \tilde{v}_1 = 0, \dots$, or $v_{(n-1)/2} = 0, \tilde{v}_{(n-1)/2} = 0$.

3 Geometric representation of polar n -complex numbers

The n -complex number $x_0 + h_1 x_1 + h_2 x_2 + \dots + h_{n-1} x_{n-1}$ can be represented by the point A of coordinates $(x_0, x_1, \dots, x_{n-1})$. If O is the origin of the n -dimensional space, the distance from the origin O to the point A of coordinates $(x_0, x_1, \dots, x_{n-1})$ has the expression

$$d^2 = x_0^2 + x_1^2 + \dots + x_{n-1}^2. \quad (20)$$

The quantity d will be called modulus of the n -complex number $u = x_0 + h_1 x_1 + h_2 x_2 + \dots + h_{n-1} x_{n-1}$. The modulus of an n -complex number u will be designated by $d = |u|$.

The exponential and trigonometric forms of the n -complex number u can be obtained conveniently in a rotated system of axes defined by a transformation which, for even n , has

the form

$$\begin{pmatrix} \xi_+ \\ \xi_- \\ \vdots \\ \xi_k \\ \eta_k \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{n}} \\ \vdots & \vdots & & \vdots & \vdots \\ \sqrt{\frac{2}{n}} & \sqrt{\frac{2}{n}} \cos \frac{2\pi k}{n} & \cdots & \sqrt{\frac{2}{n}} \cos \frac{2\pi(n-2)k}{n} & \sqrt{\frac{2}{n}} \cos \frac{2\pi(n-1)k}{n} \\ 0 & \sqrt{\frac{2}{n}} \sin \frac{2\pi k}{n} & \cdots & \sqrt{\frac{2}{n}} \sin \frac{2\pi(n-2)k}{n} & \sqrt{\frac{2}{n}} \sin \frac{2\pi(n-1)k}{n} \\ \vdots & \vdots & & \vdots & \vdots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ \vdots \\ x_{n-1} \end{pmatrix}, \quad (21)$$

where $k = 1, 2, \dots, n/2 - 1$. For odd n the rotation of the axes is described by the relations

$$\begin{pmatrix} \xi_+ \\ \xi_1 \\ \eta_1 \\ \vdots \\ \xi_k \\ \eta_k \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \sqrt{\frac{2}{n}} & \sqrt{\frac{2}{n}} \cos \frac{2\pi}{n} & \cdots & \sqrt{\frac{2}{n}} \cos \frac{2\pi(n-1)}{n} \\ 0 & \sqrt{\frac{2}{n}} \sin \frac{2\pi}{n} & \cdots & \sqrt{\frac{2}{n}} \sin \frac{2\pi(n-1)}{n} \\ \vdots & \vdots & & \vdots \\ \sqrt{\frac{2}{n}} & \sqrt{\frac{2}{n}} \cos \frac{2\pi k}{n} & \cdots & \sqrt{\frac{2}{n}} \cos \frac{2\pi(n-1)k}{n} \\ 0 & \sqrt{\frac{2}{n}} \sin \frac{2\pi k}{n} & \cdots & \sqrt{\frac{2}{n}} \sin \frac{2\pi(n-1)k}{n} \\ \vdots & \vdots & & \vdots \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ \vdots \\ \vdots \\ x_{n-1} \end{pmatrix}, \quad (22)$$

where $k = 0, 1, \dots, (n-1)/2$. The lines of the matrices in Eqs. (21) or (22) give the components of the n basis vectors of the new system of axes. These vectors have unit length and are orthogonal to each other. By comparing Eqs. (14)-(17) and (21)-(22) it can be seen that

$$v_+ = \sqrt{n}\xi_+, v_- = \sqrt{n}\xi_-, v_k = \sqrt{\frac{n}{2}}\xi_k, \tilde{v}_k = \sqrt{\frac{n}{2}}\eta_k, \quad (23)$$

i.e. the two sets of variables differ only by scale factors.

The sum of the squares of the variables v_k, \tilde{v}_k is, for even n ,

$$\sum_{k=1}^{n/2-1} (v_k^2 + \tilde{v}_k^2) = \frac{n-2}{2}(x_0^2 + \cdots + x_{n-1}^2) - 2(x_0x_2 + \cdots + x_{n-4}x_{n-2} + x_1x_3 + \cdots + x_{n-3}x_{n-1}), \quad (24)$$

and for odd n the sum is

$$\sum_{k=1}^{(n-1)/2} (v_k^2 + \tilde{v}_k^2) = \frac{n-1}{2}(x_0^2 + \cdots + x_{n-1}^2) - (x_0x_1 + \cdots + x_{n-2}x_{n-1}). \quad (25)$$

The relation (24) has been obtained with the aid of the identity, valid for even n ,

$$\sum_{k=1}^{n/2-1} \cos \frac{2\pi pk}{n} = \begin{cases} -1, & \text{for even } p, \\ 0, & \text{for odd } p. \end{cases} \quad (26)$$

The relation (25) has been obtained with the aid of the identity, valid for odd values of n ,

$$\sum_{k=1}^{(n-1)/2} \cos \frac{2\pi pk}{n} = -\frac{1}{2}. \quad (27)$$

From Eq. (24) it results that, for even n ,

$$d^2 = \frac{1}{n}v_+^2 + \frac{1}{n}v_-^2 + \frac{2}{n} \sum_{k=1}^{n/2-1} \rho_k^2, \quad (28)$$

and from Eq. (25) it results that, for odd n ,

$$d^2 = \frac{1}{n}v_+^2 + \frac{2}{n} \sum_{k=1}^{(n-1)/2} \rho_k^2. \quad (29)$$

The relations (28) and (29) show that the square of the distance d , Eq. (20), is the sum of the squares of the projections $v_+/\sqrt{n}, \rho_k\sqrt{2/n}$ and, for even n , of the square of v_-/\sqrt{n} . This is consistent with the fact that the transformation in Eqs. (21) or (22) is unitary.

The position of the point A of coordinates $(x_0, x_1, \dots, x_{n-1})$ can be also described with the aid of the distance d , Eq. (20), and of $n-1$ angles defined further. Thus, in the plane of the axes v_k, \tilde{v}_k , the radius ρ_k and the azimuthal angle ϕ_k can be introduced by the relations

$$\rho_k^2 = v_k^2 + \tilde{v}_k^2, \cos \phi_k = v_k/\rho_k, \sin \phi_k = \tilde{v}_k/\rho_k, 0 \leq \phi_k < 2\pi, \quad (30)$$

so that there are $[(n-1)/2]$ azimuthal angles. If the projection of the point A on the plane of the axes v_k, \tilde{v}_k is A_k , and the projection of the point A on the 4-dimensional space defined by the axes $v_1, \tilde{v}_1, v_k, \tilde{v}_k$ is A_{1k} , the angle ψ_{k-1} between the line OA_{1k} and the 2-dimensional plane defined by the axes v_k, \tilde{v}_k is

$$\tan \psi_{k-1} = \rho_1/\rho_k, \quad (31)$$

where $0 \leq \psi_k \leq \pi/2, k = 2, \dots, [(n-1)/2]$, so that there are $[(n-3)/2]$ planar angles. Moreover, there is a polar angle θ_+ , which can be defined as the angle between the line OA_{1+} and the axis v_+ , where A_{1+} is the projection of the point A on the 3-dimensional space generated by the axes v_1, \tilde{v}_1, v_+ ,

$$\tan \theta_+ = \frac{\sqrt{2}\rho_1}{v_+}, \quad (32)$$

where $0 \leq \theta_+ \leq \pi$, and in an even number of dimensions n there is also a polar angle θ_- , which can be defined as the angle between the line OA_{1-} and the axis v_- , where A_{1-} is the projection of the point A on the 3-dimensional space generated by the axes v_1, \tilde{v}_1, v_- ,

$$\tan \theta_- = \frac{\sqrt{2}\rho_1}{v_-}, \quad (33)$$

where $0 \leq \theta_- \leq \pi$. In Eqs. (32) and (33), the factor $\sqrt{2}$ appears from the ratio of the normalization factors in Eq. (23). Thus, the position of the point A is described, in an even number of dimensions, by the distance d , by $n/2 - 1$ azimuthal angles, by $n/2 - 2$ planar angles, and by 2 polar angles. In an odd number of dimensions, the position of the point A is described by $(n-1)/2$ azimuthal angles, by $(n-3)/2$ planar angles, and by 1 polar angle. These angles are shown in Fig. 2.

The variables ρ_k can be expressed in terms of d and the planar angles ψ_k as

$$\rho_k = \frac{\rho_1}{\tan \psi_{k-1}}, \quad (34)$$

for $k = 2, \dots, [(n-1)/2]$, where, for even n ,

$$\rho_1^2 = \frac{nd^2}{2} \left(\frac{1}{\tan^2 \theta_+} + \frac{1}{\tan^2 \theta_-} + 1 + \frac{1}{\tan^2 \psi_1} + \frac{1}{\tan^2 \psi_2} + \dots + \frac{1}{\tan^2 \psi_{n/2-2}} \right)^{-1}, \quad (35)$$

and for odd n

$$\rho_1^2 = \frac{nd^2}{2} \left(\frac{1}{\tan^2 \theta_+} + 1 + \frac{1}{\tan^2 \psi_1} + \frac{1}{\tan^2 \psi_2} + \dots + \frac{1}{\tan^2 \psi_{(n-3)/2}} \right)^{-1}. \quad (36)$$

If $u' = x'_0 + h_1 x'_1 + h_2 x'_2 + \dots + h_{n-1} x'_{n-1}$, $u'' = x''_0 + h_1 x''_1 + h_2 x''_2 + \dots + h_{n-1} x''_{n-1}$ are n -complex numbers of parameters $v'_+, v'_-, \rho'_k, \theta'_+, \theta'_-, \psi'_k, \phi'_k$ and respectively $v''_+, v''_-, \rho''_k, \theta''_+, \theta''_-, \psi''_k, \phi''_k$, then the parameters $v_+, v_-, \rho_k, \theta_+, \theta_-, \psi_k, \phi_k$ of the product n -complex number $u = u' u''$ are given by

$$v_+ = v'_+ v''_+, \quad (37)$$

$$\rho_k = \rho'_k \rho''_k, \quad (38)$$

for $k = 1, \dots, [(n-1)/2]$,

$$\tan \theta_+ = \frac{1}{\sqrt{2}} \tan \theta'_+ \tan \theta''_+, \quad (39)$$

$$\tan \psi_k = \tan \psi'_k \tan \psi''_k, \quad (40)$$

for $k = 1, \dots, [(n-3)/2]$,

$$\phi_k = \phi'_k + \phi''_k, \quad (41)$$

for $k = 1, \dots, [(n-1)/2]$, and, if n is even,

$$v_- = v'_- v''_-, \quad (42)$$

$$\tan \theta_- = \frac{1}{\sqrt{2}} \tan \theta'_- \tan \theta''_-. \quad (43)$$

The Eqs. (37) and (42) can be checked directly, and Eqs. (38)-(41) and (43) are a consequence of the relations

$$v_k = v'_k v''_k - \tilde{v}'_k \tilde{v}''_k, \quad \tilde{v}_k = v'_k \tilde{v}''_k + \tilde{v}'_k v''_k, \quad (44)$$

and of the corresponding relations of definition. Then the product ν in Eqs. (18) and (19) has the property that

$$\nu = \nu' \nu'' \quad (45)$$

and, if $\nu' > 0, \nu'' > 0$, the amplitude ρ defined in Eq. (11) has the property that

$$\rho = \rho' \rho''. \quad (46)$$

The fact that the amplitude of the product is equal to the product of the amplitudes, as written in Eq. (46), can be demonstrated also by using a representation of the n -complex numbers by matrices, in which the n -complex number $u = x_0 + h_1 x_1 + h_2 x_2 + \dots + h_{n-1} x_{n-1}$ is represented by the matrix

$$U = \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & x_1 & \cdots & x_{n-2} \\ x_{n-2} & x_{n-1} & x_0 & \cdots & x_{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_1 & x_2 & x_3 & \cdots & x_0 \end{pmatrix}. \quad (47)$$

The product $u = u' u''$ is represented by the matrix multiplication $U = U' U''$. The relation (45) is then a consequence of the fact the determinant of the product of matrices is equal to the product of the determinants of the factor matrices. The use of the representation of the

n-complex numbers with matrices provides an alternative demonstration of the fact that the product of n-complex numbers is associative, as stated in Eq. (5).

According to Eqs. (37), (38), (42), (28) and (29), the modulus of the product uu' is, for even n ,

$$|uu'|^2 = \frac{1}{n}(v_+v'_+)^2 + \frac{1}{n}(v_-v'_-)^2 + \frac{2}{n} \sum_{k=1}^{n/2-1} (\rho_k \rho'_k)^2, \quad (48)$$

and for odd n

$$|uu'|^2 = \frac{1}{n}(v_+v'_+)^2 + \frac{2}{n} \sum_{k=1}^{(n-1)/2} (\rho_k \rho'_k)^2. \quad (49)$$

Thus, if the product of two n-complex numbers is zero, $uu' = 0$, then $v_+v'_+ = 0, \rho_k \rho'_k = 0, k = 1, \dots, [(n-1)/2]$ and, if n is even, $v_-v'_- = 0$. This means that either $u = 0$, or $u' = 0$, or u, u' belong to orthogonal hypersurfaces in such a way that the afore-mentioned products of components should be equal to zero.

4 The polar n-dimensional cosexponential functions

The exponential function of the n-complex variable u can be defined by the series

$$\exp u = 1 + u + u^2/2! + u^3/3! + \dots \quad (50)$$

It can be checked by direct multiplication of the series that

$$\exp(u + u') = \exp u \cdot \exp u'. \quad (51)$$

If $u = x_0 + h_1x_1 + h_2x_2 + \dots + h_{n-1}x_{n-1}$, then $\exp u$ can be calculated as $\exp u = \exp x_0 \cdot \exp(h_1x_1) \cdots \exp(h_{n-1}x_{n-1})$.

It can be seen with the aid of the representation in Fig. 1 that

$$h_k^{n+p} = h_k^p, \quad p \text{ integer}, \quad (52)$$

for $k = 1, \dots, n-1$. Then $e^{h_k y}$ can be written as

$$e^{h_k y} = \sum_{p=0}^{n-1} h_{kp-n[kp/n]} g_{np}(y), \quad (53)$$

where the expression of the functions g_{nk} , which will be called polar cosexponential functions in n dimensions, is

$$g_{nk}(y) = \sum_{p=0}^{\infty} y^{k+pn} / (k+pn)!, \quad (54)$$

for $k = 0, 1, \dots, n-1$.

If n is even, the polar cosexponential functions of even index k are even functions, $g_{n,2p}(-y) = g_{n,2p}(y)$, $p = 0, 1, \dots, n/2 - 1$, and the polar cosexponential functions of odd index are odd functions, $g_{n,2p+1}(-y) = -g_{n,2p+1}(y)$, $p = 0, 1, \dots, n/2 - 1$. For odd values of n , the polar cosexponential functions do not have a definite parity. It can be checked that

$$\sum_{k=0}^{n-1} g_{nk}(y) = e^y \quad (55)$$

and, for even n ,

$$\sum_{k=0}^{n-1} (-1)^k g_{nk}(y) = e^{-y}. \quad (56)$$

The expression of the polar n -dimensional cosexponential functions is

$$g_{nk}(y) = \frac{1}{n} \sum_{l=0}^{n-1} \exp \left[y \cos \left(\frac{2\pi l}{n} \right) \right] \cos \left[y \sin \left(\frac{2\pi l}{n} \right) - \frac{2\pi kl}{n} \right], \quad (57)$$

for $k = 0, 1, \dots, n-1$. In order to check that the function in Eq. (57) has the series expansion written in Eq. (54), the right-hand side of Eq. (57) will be written as

$$g_{nk}(y) = \frac{1}{n} \sum_{l=0}^{n-1} \operatorname{Re} \left\{ \exp \left[\left(\cos \frac{2\pi l}{n} + i \sin \frac{2\pi l}{n} \right) y - i \frac{2\pi kl}{n} \right] \right\}, \quad (58)$$

for $k = 0, 1, \dots, n-1$, where $\operatorname{Re}(a + ib) = a$, with a and b real numbers. The part of the exponential depending on y can be expanded in a series,

$$g_{nk}(y) = \frac{1}{n} \sum_{p=0}^{\infty} \sum_{l=0}^{n-1} \operatorname{Re} \left\{ \frac{1}{p!} \exp \left[i \frac{2\pi l}{n} (p - k) \right] y^p \right\}, \quad (59)$$

for $k = 0, 1, \dots, n-1$. The expression of $g_{nk}(y)$ becomes

$$g_{nk}(y) = \frac{1}{n} \sum_{p=0}^{\infty} \sum_{l=0}^{n-1} \left\{ \frac{1}{p!} \cos \left[\frac{2\pi l}{n} (p - k) \right] y^p \right\}, \quad (60)$$

for $k = 0, 1, \dots, n-1$ and, since

$$\frac{1}{n} \sum_{l=0}^{n-1} \cos \frac{2\pi l}{n} (p - k) = \begin{cases} 1, & \text{if } p - k \text{ is a multiple of } n, \\ 0, & \text{otherwise,} \end{cases} \quad (61)$$

this yields indeed the expansion in Eq. (54).

It can be shown from Eq. (57) that

$$\sum_{k=0}^{n-1} g_{nk}^2(y) = \frac{1}{n} \sum_{l=0}^{n-1} \exp \left[2y \cos \left(\frac{2\pi l}{n} \right) \right]. \quad (62)$$

It can be seen that the right-hand side of Eq. (62) does not contain oscillatory terms. If n is a multiple of 4, it can be shown by replacing y by iy in Eq. (62) that

$$\sum_{k=0}^{n-1} (-1)^k g_{nk}^2(y) = \frac{2}{n} \left\{ 1 + \cos 2y + \sum_{l=1}^{n/4-1} \cos \left[2y \cos \left(\frac{2\pi l}{n} \right) \right] \right\}, \quad (63)$$

which does not contain exponential terms.

Addition theorems for the polar n -dimensional cosexponential functions can be obtained from the relation $\exp h_1(y+z) = \exp h_1 y \cdot \exp h_1 z$, by substituting the expression of the exponentials as given in Eq. (53) for $k=1$, $e^{h_1 y} = g_{n0}(y) + h_1 g_{n1}(y) + \cdots + h_{n-1} g_{n,n-1}(y)$,

$$\begin{aligned} g_{nk}(y+z) &= g_{n0}(y)g_{nk}(z) + g_{n1}(y)g_{n,k-1}(z) + \cdots + g_{nk}(y)g_{n0}(z) \\ &\quad + g_{n,k+1}(y)g_{n,n-1}(z) + g_{n,k+2}(y)g_{n,n-2}(z) + \cdots + g_{n,n-1}(y)g_{n,k+1}(z), \end{aligned} \quad (64)$$

where $k=0, 1, \dots, n-1$. For $y=z$ the relations (64) take the form

$$\begin{aligned} g_{nk}(2y) &= g_{n0}(y)g_{nk}(y) + g_{n1}(y)g_{n,k-1}(y) + \cdots + g_{nk}(y)g_{n0}(y) \\ &\quad + g_{n,k+1}(y)g_{n,n-1}(y) + g_{n,k+2}(y)g_{n,n-2}(y) + \cdots + g_{n,n-1}(y)g_{n,k+1}(y), \end{aligned} \quad (65)$$

where $k=0, 1, \dots, n-1$. For $y=-z$ the relations (64) and (54) yield

$$g_{n0}(y)g_{n0}(-y) + g_{n1}(y)g_{n,n-1}(-y) + g_{n2}(y)g_{n,n-2}(-y) + \cdots + g_{n,n-1}(y)g_{n1}(-y) = 1, \quad (66)$$

$$\begin{aligned} &g_{n0}(y)g_{nk}(-y) + g_{n1}(y)g_{n,k-1}(-y) + \cdots + g_{nk}(y)g_{n0}(-y) \\ &\quad + g_{n,k+1}(y)g_{n,n-1}(-y) + g_{n,k+2}(y)g_{n,n-2}(-y) + \cdots + g_{n,n-1}(y)g_{n,k+1}(-y) = 0, \end{aligned} \quad (67)$$

for $k=1, \dots, n-1$.

From Eq. (53) it can be shown, for natural numbers l , that

$$\left(\sum_{p=0}^{n-1} h_{kp-n[kp/n]} g_{np}(y) \right)^l = \sum_{p=0}^{n-1} h_{kp-n[kp/n]} g_{np}(ly), \quad (68)$$

where $k = 0, 1, \dots, n-1$. For $k = 1$ the relation (68) is

$$\{g_{n0}(y) + h_1 g_{n1}(y) + \dots + h_{n-1} g_{n,n-1}(y)\}^l = g_{n0}(ly) + h_1 g_{n1}(ly) + \dots + h_{n-1} g_{n,n-1}(ly). \quad (69)$$

If

$$a_k = \sum_{p=0}^{n-1} g_{np}(y) \cos\left(\frac{2\pi kp}{n}\right), \quad (70)$$

for $k = 0, 1, \dots, n-1$, and

$$b_k = \sum_{p=0}^{n-1} g_{np}(y) \sin\left(\frac{2\pi kp}{n}\right), \quad (71)$$

for $k = 1, \dots, n-1$, where $g_{nk}(y)$ are the polar cosexponential functions in Eq. (57), it can be shown that

$$a_k = \exp\left[y \cos\left(\frac{2\pi k}{n}\right)\right] \cos\left[y \sin\left(\frac{2\pi k}{n}\right)\right], \quad (72)$$

where $k = 0, 1, \dots, n-1$,

$$b_k = \exp\left[y \cos\left(\frac{2\pi k}{n}\right)\right] \sin\left[y \sin\left(\frac{2\pi k}{n}\right)\right], \quad (73)$$

where $k = 1, \dots, n-1$. If

$$G_k^2 = a_k^2 + b_k^2, \quad (74)$$

for $k = 1, \dots, n-1$, then from Eqs. (72) and (73) it results that

$$G_k^2 = \exp\left[2y \cos\left(\frac{2\pi k}{n}\right)\right], \quad (75)$$

where $k = 1, \dots, n-1$. If

$$G_+ = g_{n0} + g_{n1} + \dots + g_{n,n-1}, \quad (76)$$

from Eq. (70) it results that $G_+ = a_0$, so that $G_+ = e^y$, and, in an even number of dimensions n , if

$$G_- = g_{n0} - g_{n1} + \dots + g_{n,n-2} - g_{n,n-1}, \quad (77)$$

from Eq. (70) it results that $G_- = a_{n/2}$, so that $G_{n/2} = e^{-y}$. Then with the aid of Eq. (26) applied for $p = 1$ it can be shown that the polar n -dimensional cosexponential functions have the property that, for even n ,

$$G_+ G_- \prod_{k=1}^{n/2-1} G_k^2 = 1, \quad (78)$$

and in an odd number of dimensions, with the aid of Eq. (27) it can be shown that

$$G_+ \prod_{k=1}^{(n-1)/2} G_k^2 = 1. \quad (79)$$

The polar n -dimensional cosexponential functions are solutions of the n^{th} -order differential equation

$$\frac{d^n \zeta}{du^n} = \zeta, \quad (80)$$

whose solutions are of the form $\zeta(u) = A_0 g_{n0}(u) + A_1 g_{n1}(u) + \dots + A_{n-1} g_{n,n-1}(u)$. It can be checked that the derivatives of the polar cosexponential functions are related by

$$\frac{dg_{n0}}{du} = g_{n,n-1}, \quad \frac{dg_{n1}}{du} = g_{n0}, \quad \dots, \quad \frac{dg_{n,n-2}}{du} = g_{n,n-3}, \quad \frac{dg_{n,n-1}}{du} = g_{n,n-2}. \quad (81)$$

5 Exponential and trigonometric forms of polar n -complex numbers

In order to obtain the exponential and trigonometric forms of n -complex numbers, a canonical base $e_+, e_-, e_1, \tilde{e}_1, \dots, e_{n/2-1}, \tilde{e}_{n/2-1}$ for the polar n -complex numbers will be introduced for even n by the relations

$$\begin{pmatrix} e_+ \\ e_- \\ \vdots \\ e_k \\ \tilde{e}_k \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} & \frac{1}{n} \\ \frac{1}{n} & -\frac{1}{n} & \dots & \frac{1}{n} & -\frac{1}{n} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{2}{n} & \frac{2}{n} \cos \frac{2\pi k}{n} & \dots & \frac{2}{n} \cos \frac{2\pi(n-2)k}{n} & \frac{2}{n} \cos \frac{2\pi(n-1)k}{n} \\ 0 & \frac{2}{n} \sin \frac{2\pi k}{n} & \dots & \frac{2}{n} \sin \frac{2\pi(n-2)k}{n} & \frac{2}{n} \sin \frac{2\pi(n-1)k}{n} \\ \vdots & \vdots & & \vdots & \vdots \end{pmatrix} \begin{pmatrix} 1 \\ h_1 \\ \vdots \\ \vdots \\ h_{n-1} \end{pmatrix}, \quad (82)$$

where $k = 1, 2, \dots, n/2 - 1$. For odd n , the canonical base $e_+, e_1, \tilde{e}_1, \dots, e_{(n-1)/2}, \tilde{e}_{(n-1)/2}$ for the polar n -complex numbers will be introduced by the relations

$$\begin{pmatrix} e_+ \\ e_1 \\ \tilde{e}_1 \\ \vdots \\ e_k \\ \tilde{e}_k \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{2}{n} & \frac{2}{n} \cos \frac{2\pi}{n} & \cdots & \frac{2}{n} \cos \frac{2\pi(n-1)}{n} \\ 0 & \frac{2}{n} \sin \frac{2\pi}{n} & \cdots & \frac{2}{n} \sin \frac{2\pi(n-1)}{n} \\ \vdots & \vdots & & \vdots \\ \frac{2}{n} & \frac{2}{n} \cos \frac{2\pi k}{n} & \cdots & \frac{2}{n} \cos \frac{2\pi(n-1)k}{n} \\ 0 & \frac{2}{n} \sin \frac{2\pi k}{n} & \cdots & \frac{2}{n} \sin \frac{2\pi(n-1)k}{n} \\ \vdots & \vdots & & \vdots \end{pmatrix} \begin{pmatrix} 1 \\ h_1 \\ h_2 \\ \vdots \\ \vdots \\ \vdots \\ h_{n-1} \end{pmatrix}, \quad (83)$$

where $k = 0, 1, \dots, (n-1)/2$.

The multiplication relations for the new bases are, for even n ,

$$\begin{aligned} e_+^2 &= e_+, \quad e_-^2 = e_-, \quad e_+ e_- = 0, \quad e_+ e_k = 0, \quad e_+ \tilde{e}_k = 0, \quad e_- e_k = 0, \quad e_- \tilde{e}_k = 0, \\ e_k^2 &= e_k, \quad \tilde{e}_k^2 = -e_k, \quad e_k \tilde{e}_k = \tilde{e}_k, \quad e_k e_l = 0, \quad e_k \tilde{e}_l = 0, \quad \tilde{e}_k \tilde{e}_l = 0, \quad k \neq l, \end{aligned} \quad (84)$$

where $k, l = 1, \dots, n/2 - 1$. For odd n the multiplication relations are

$$\begin{aligned} e_+^2 &= e_+, \quad e_+ e_k = 0, \quad e_+ \tilde{e}_k = 0, \\ e_k^2 &= e_k, \quad \tilde{e}_k^2 = -e_k, \quad e_k \tilde{e}_k = \tilde{e}_k, \quad e_k e_l = 0, \quad e_k \tilde{e}_l = 0, \quad \tilde{e}_k \tilde{e}_l = 0, \quad k \neq l, \end{aligned} \quad (85)$$

where $k, l = 1, \dots, (n-1)/2$. The moduli of the new bases are

$$|e_+| = \frac{1}{\sqrt{n}}, \quad |e_-| = \frac{1}{\sqrt{n}}, \quad |e_k| = \sqrt{\frac{2}{n}}, \quad |\tilde{e}_k| = \sqrt{\frac{2}{n}}. \quad (86)$$

It can be shown that, for even n ,

$$x_0 + h_1 x_1 + \cdots + h_{n-1} x_{n-1} = e_+ v_+ + e_- v_- + \sum_{k=1}^{n/2-1} (e_k v_k + \tilde{e}_k \tilde{v}_k), \quad (87)$$

and for odd n

$$x_0 + h_1 x_1 + \cdots + h_{n-1} x_{n-1} = e_+ v_+ + \sum_{k=1}^{(n-1)/2} (e_k v_k + \tilde{e}_k \tilde{v}_k). \quad (88)$$

The relations (87), (88) give the canonical form of a polar n -complex number.

Using the properties of the bases in Eqs. (84) and (85) it can be shown that

$$\exp(\tilde{e}_k \phi_k) = 1 - e_k + e_k \cos \phi_k + \tilde{e}_k \sin \phi_k, \quad (89)$$

$$\exp(e_k \ln \rho_k) = 1 - e_k + e_k \rho_k, \quad (90)$$

$$\exp(e_+ \ln v_+) = 1 - e_+ + e_+ v_+ \quad (91)$$

and, for even n ,

$$\exp(e_- \ln v_-) = 1 - e_- + e_- v_-. \quad (92)$$

In Eq. (91), $\ln v_+$ exists as a real function provided that $v_+ = x_0 + x_1 + \cdots + x_{n-1} > 0$, which means that $0 < \theta_+ < \pi/2$, and for even n , $\ln v_-$ exists in Eq. (92) as a real function provided that $v_- = x_0 - x_1 + \cdots + x_{n-2} - x_{n-1} > 0$, which means that $0 < \theta_- < \pi/2$. By multiplying the relations (89)-(92) it results, for even n , that

$$\exp \left[e_+ \ln v_+ + e_- \ln v_- + \sum_{k=1}^{n/2-1} (e_k \ln \rho_k + \tilde{e}_k \phi_k) \right] = e_+ v_+ + e_- v_- + \sum_{k=1}^{n/2-1} (e_k v_k + \tilde{e}_k \tilde{v}_k), \quad (93)$$

where the fact has been used that

$$e_+ + e_- + \sum_{k=1}^{n/2-1} e_k = 1, \quad (94)$$

the latter relation being a consequence of Eqs. (82) and (26). Similarly, by multiplying the relations (89)-(91) it results, for odd n , that

$$\exp \left[e_+ \ln v_+ + \sum_{k=1}^{(n-1)/2} (e_k \ln \rho_k + \tilde{e}_k \phi_k) \right] = e_+ v_+ + \sum_{k=1}^{(n-1)/2} (e_k v_k + \tilde{e}_k \tilde{v}_k), \quad (95)$$

where the fact has been used that

$$e_+ + \sum_{k=1}^{(n-1)/2} e_k = 1, \quad (96)$$

the latter relation being a consequence of Eqs. (83) and (27).

By comparing Eqs. (87) and (93), it can be seen that, for even n ,

$$x_0 + h_1 x_1 + \cdots + h_{n-1} x_{n-1} = \exp \left[e_+ \ln v_+ + e_- \ln v_- + \sum_{k=1}^{n/2-1} (e_k \ln \rho_k + \tilde{e}_k \phi_k) \right], \quad (97)$$

and by comparing Eqs. (88) and (95), it can be seen that, for odd n ,

$$x_0 + h_1 x_1 + \cdots + h_{n-1} x_{n-1} = \exp \left[e_+ \ln v_+ + \sum_{k=1}^{(n-1)/2} (e_k \ln \rho_k + \tilde{e}_k \phi_k) \right]. \quad (98)$$

Using the expression of the bases in Eqs. (82) and (83) yields, for even values of n , the exponential form of the n -complex number $u = x_0 + h_1x_1 + \cdots + h_{n-1}x_{n-1}$ as

$$u = \rho \exp \left\{ \sum_{p=1}^{n-1} h_p \left[\frac{1}{n} \ln \frac{\sqrt{2}}{\tan \theta_+} + \frac{(-1)^p}{n} \ln \frac{\sqrt{2}}{\tan \theta_-} - \frac{2}{n} \sum_{k=2}^{n/2-1} \cos \left(\frac{2\pi kp}{n} \right) \ln \tan \psi_{k-1} \right] + \sum_{k=1}^{n/2-1} \tilde{e}_k \phi_k \right\}, \quad (99)$$

where ρ is the amplitude defined in Eq. (11), which for even n has according to Eq. (18) the expression

$$\rho = \left(v_+ v_- \rho_1^2 \cdots \rho_{n/2-1}^2 \right)^{1/n}. \quad (100)$$

For odd values of n , the exponential form of the n -complex number u is

$$u = \rho \exp \left\{ \sum_{p=1}^{n-1} h_p \left[\frac{1}{n} \ln \frac{\sqrt{2}}{\tan \theta_+} - \frac{2}{n} \sum_{k=2}^{(n-1)/2} \cos \left(\frac{2\pi kp}{n} \right) \ln \tan \psi_{k-1} \right] + \sum_{k=1}^{(n-1)/2} \tilde{e}_k \phi_k \right\}, \quad (101)$$

where for odd n , ρ has according to Eq. (19) the expression

$$\rho = \left(v_+ \rho_1^2 \cdots \rho_{(n-1)/2}^2 \right)^{1/n}. \quad (102)$$

It can be checked with the aid of Eq. (89) that the n -complex number u can also be written, for even n , as

$$x_0 + h_1x_1 + \cdots + h_{n-1}x_{n-1} = \left(e_+ v_+ + e_- v_- + \sum_{k=1}^{n/2-1} e_k \rho_k \right) \exp \left(\sum_{k=1}^{n/2-1} \tilde{e}_k \phi_k \right), \quad (103)$$

and for odd n , as

$$x_0 + h_1x_1 + \cdots + h_{n-1}x_{n-1} = \left(e_+ v_+ + \sum_{k=1}^{(n-1)/2} e_k \rho_k \right) \exp \left(\sum_{k=1}^{(n-1)/2} \tilde{e}_k \phi_k \right). \quad (104)$$

Writing in Eqs. (103) and (104) the radius ρ_1 , Eqs. (35) and (36), as a factor and expressing the variables in terms of the polar and planar angles with the aid of Eqs. (31)-(33) yields the trigonometric form of the n -complex number u , for even n , as

$$u = d \left(\frac{n}{2} \right)^{1/2} \left(\frac{1}{\tan^2 \theta_+} + \frac{1}{\tan^2 \theta_-} + 1 + \frac{1}{\tan^2 \psi_1} + \frac{1}{\tan^2 \psi_2} + \cdots + \frac{1}{\tan^2 \psi_{n/2-2}} \right)^{-1/2} \left(\frac{e_+ \sqrt{2}}{\tan \theta_+} + \frac{e_- \sqrt{2}}{\tan \theta_-} + e_1 + \sum_{k=2}^{n/2-1} \frac{e_k}{\tan \psi_{k-1}} \right) \exp \left(\sum_{k=1}^{n/2-1} \tilde{e}_k \phi_k \right), \quad (105)$$

and for odd n as

$$u = d \left(\frac{n}{2} \right)^{1/2} \left(\frac{1}{\tan^2 \theta_+} + 1 + \frac{1}{\tan^2 \psi_1} + \frac{1}{\tan^2 \psi_2} + \cdots + \frac{1}{\tan^2 \psi_{(n-3)/2}} \right)^{-1/2} \left(\frac{e_+ \sqrt{2}}{\tan \theta_+} + e_1 + \sum_{k=2}^{(n-1)/2} \frac{e_k}{\tan \psi_{k-1}} \right) \exp \left(\sum_{k=1}^{(n-1)/2} \tilde{e}_k \phi_k \right). \quad (106)$$

In Eqs. (105) and 106), the n -complex number u , written in trigonometric form, is the product of the modulus d , of a part depending on the polar and planar angles $\theta_+, \theta_-, \psi_1, \dots, \psi_{[(n-3)/2]}$, and of a factor depending on the azimuthal angles $\phi_1, \dots, \phi_{[(n-1)/2]}$. Although the modulus of a product of n -complex numbers is not equal in general to the product of the moduli of the factors, it can be checked that the modulus of the factor in Eq. (105) is

$$\left| \frac{e_+ \sqrt{2}}{\tan \theta_+} + \frac{e_- \sqrt{2}}{\tan \theta_-} + e_1 + \sum_{k=2}^{n/2-1} \frac{e_k}{\tan \psi_{k-1}} \right| = \left(\frac{2}{n} \right)^{1/2} \left(\frac{1}{\tan^2 \theta_+} + \frac{1}{\tan^2 \theta_-} + 1 + \frac{1}{\tan^2 \psi_1} + \frac{1}{\tan^2 \psi_2} + \cdots + \frac{1}{\tan^2 \psi_{n/2-2}} \right)^{1/2}, \quad (107)$$

and the modulus of the factor in Eq. (106) is

$$\left| \frac{e_+ \sqrt{2}}{\tan \theta_+} + e_1 + \sum_{k=2}^{(n-1)/2} \frac{e_k}{\tan \psi_{k-1}} \right| = \left(\frac{2}{n} \right)^{1/2} \left(\frac{1}{\tan^2 \theta_+} + 1 + \frac{1}{\tan^2 \psi_1} + \frac{1}{\tan^2 \psi_2} + \cdots + \frac{1}{\tan^2 \psi_{(n-3)/2}} \right)^{1/2}. \quad (108)$$

Moreover, it can be checked that

$$\left| \exp \left[\sum_{k=1}^{[(n-1)/2]} \tilde{e}_k \phi_k \right] \right| = 1. \quad (109)$$

The modulus d in Eqs. (105) and (106) can be expressed in terms of the amplitude ρ , for even n , as

$$d = \rho \frac{2^{(n-2)/2n}}{\sqrt{n}} \left(\tan \theta_+ \tan \theta_- \tan^2 \psi_1 \cdots \tan^2 \psi_{n/2-2} \right)^{1/n} \left(\frac{1}{\tan^2 \theta_+} + \frac{1}{\tan^2 \theta_-} + 1 + \frac{1}{\tan^2 \psi_1} + \frac{1}{\tan^2 \psi_2} + \cdots + \frac{1}{\tan^2 \psi_{n/2-2}} \right)^{1/2}, \quad (110)$$

and for odd n as

$$d = \rho \frac{2^{(n-1)/2n}}{\sqrt{n}} \left(\tan \theta_+ \tan^2 \psi_1 \cdots \tan^2 \psi_{(n-3)/2} \right)^{1/n} \left(\frac{1}{\tan^2 \theta_+} + 1 + \frac{1}{\tan^2 \psi_1} + \frac{1}{\tan^2 \psi_2} + \cdots + \frac{1}{\tan^2 \psi_{(n-3)/2}} \right)^{1/2}. \quad (111)$$

6 Elementary functions of a polar n-complex variable

The logarithm u_1 of the n-complex number u , $u_1 = \ln u$, can be defined as the solution of the equation

$$u = e^{u_1}. \quad (112)$$

For even n the relation (93) shows that $\ln u$ exists as an n-complex function with real components if $v_+ = x_0 + x_1 + \cdots + x_{n-1} > 0$ and $v_- = x_0 - x_1 + \cdots + x_{n-2} - x_{n-1} > 0$, which means that $0 < \theta_+ < \pi/2, 0 < \theta_- < \pi/2$. For odd n the relation (95) shows that $\ln u$ exists as an n-complex function with real components if $v_+ = x_0 + x_1 + \cdots + x_{n-1} > 0$, which means that $0 < \theta_+ < \pi/2$. The expression of the logarithm, obtained from Eqs. (97) and (98), is, for even n ,

$$\ln u = e_+ \ln v_+ + e_- \ln v_- + \sum_{k=1}^{n/2-1} (e_k \ln \rho_k + \tilde{e}_k \phi_k), \quad (113)$$

and for odd n the expression is

$$\ln u = e_+ \ln v_+ + \sum_{k=1}^{(n-1)/2} (e_k \ln \rho_k + \tilde{e}_k \phi_k). \quad (114)$$

An expression of the logarithm depending on the amplitude ρ can be obtained from the exponential forms in Eqs. (99) and (101), for even n , as

$$\begin{aligned} \ln u = \ln \rho + \sum_{p=1}^{n-1} h_p \left[\frac{1}{n} \ln \frac{\sqrt{2}}{\tan \theta_+} + \frac{(-1)^p}{n} \ln \frac{\sqrt{2}}{\tan \theta_-} - \frac{2}{n} \sum_{k=2}^{n/2-1} \cos \left(\frac{2\pi kp}{n} \right) \ln \tan \psi_{k-1} \right] \\ + \sum_{k=1}^{n/2-1} \tilde{e}_k \phi_k, \end{aligned} \quad (115)$$

and for odd n as

$$\ln u = \ln \rho + \sum_{p=1}^{n-1} h_p \left[\frac{1}{n} \ln \frac{\sqrt{2}}{\tan \theta_+} - \frac{2}{n} \sum_{k=2}^{(n-1)/2} \cos \left(\frac{2\pi kp}{n} \right) \ln \tan \psi_{k-1} \right] + \sum_{k=1}^{(n-1)/2} \tilde{e}_k \phi_k. \quad (116)$$

The function $\ln u$ is multivalued because of the presence of the terms $\tilde{e}_k \phi_k$. It can be inferred from Eqs. (37)-(43) and (46) that

$$\ln(uu') = \ln u + \ln u', \quad (117)$$

up to integer multiples of $2\pi\tilde{e}_k, k = 1, \dots, [(n-1)/2]$.

The power function u^m can be defined for real values of m as

$$u^m = e^{m \ln u}. \quad (118)$$

Using the expression of $\ln u$ in Eqs. (113) and (114) yields, for even values of n ,

$$u^m = e_+ v_+^m + e_- v_-^m + \sum_{k=1}^{n/2-1} \rho_k^m (e_k \cos m\phi_k + \tilde{e}_k \sin m\phi_k), \quad (119)$$

and for odd values of n

$$u^m = e_+ v_+^m + \sum_{k=1}^{(n-1)/2} \rho_k^m (e_k \cos m\phi_k + \tilde{e}_k \sin m\phi_k). \quad (120)$$

For integer values of m , the relations (119) and (120) are valid for any x_0, \dots, x_{n-1} . The power function is multivalued unless m is an integer. For integer m , it can be inferred from Eq. (117) that

$$(uu')^m = u^m u'^m. \quad (121)$$

The trigonometric functions $\cos u$ and $\sin u$ of an n -complex variable u are defined by the series

$$\cos u = 1 - u^2/2! + u^4/4! + \dots, \quad (122)$$

$$\sin u = u - u^3/3! + u^5/5! + \dots. \quad (123)$$

It can be checked by series multiplication that the usual addition theorems hold for the n -complex numbers u, u' ,

$$\cos(u + u') = \cos u \cos u' - \sin u \sin u', \quad (124)$$

$$\sin(u + u') = \sin u \cos u' + \cos u \sin u'. \quad (125)$$

In order to obtain expressions for the trigonometric functions of n -complex variables, these will be expressed with the aid of the imaginary unit i as

$$\cos u = \frac{1}{2}(e^{iu} + e^{-iu}), \quad \sin u = \frac{1}{2i}(e^{iu} - e^{-iu}). \quad (126)$$

The imaginary unit i is used for the convenience of notations, and it does not appear in the final results. The validity of Eq. (126) can be checked by comparing the series for the

two sides of the relations. Since the expression of the exponential function $e^{h_k y}$ in terms of the units $1, h_1, \dots, h_{n-1}$ given in Eq. (53) depends on the polar cosexponential functions $g_{np}(y)$, the expression of the trigonometric functions will depend on the functions $g_{p+}^{(c)}(y) = (1/2)[g_{np}(iy) + g_{np}(-iy)]$ and $g_{p-}^{(c)}(y) = (1/2i)[g_{np}(iy) - g_{np}(-iy)]$,

$$\cos(h_k y) = \sum_{p=0}^{n-1} h_{kp-n[kp/n]} g_{p+}^{(c)}(y), \quad (127)$$

$$\sin(h_k y) = \sum_{p=0}^{n-1} h_{kp-n[kp/n]} g_{p-}^{(c)}(y), \quad (128)$$

where

$$g_{p+}^{(c)}(y) = \frac{1}{n} \sum_{l=0}^{n-1} \left\{ \cos \left[y \cos \left(\frac{2\pi l}{n} \right) \right] \cosh \left[y \sin \left(\frac{2\pi l}{n} \right) \right] \cos \left(\frac{2\pi l p}{n} \right) - \sin \left[y \cos \left(\frac{2\pi l}{n} \right) \right] \sinh \left[y \sin \left(\frac{2\pi l}{n} \right) \right] \sin \left(\frac{2\pi l p}{n} \right) \right\}, \quad (129)$$

$$g_{p-}^{(c)}(y) = \frac{1}{n} \sum_{l=0}^{n-1} \left\{ \sin \left[y \cos \left(\frac{2\pi l}{n} \right) \right] \cosh \left[y \sin \left(\frac{2\pi l}{n} \right) \right] \cos \left(\frac{2\pi l p}{n} \right) + \cos \left[y \cos \left(\frac{2\pi l}{n} \right) \right] \sinh \left[y \sin \left(\frac{2\pi l}{n} \right) \right] \sin \left(\frac{2\pi l p}{n} \right) \right\}. \quad (130)$$

The hyperbolic functions $\cosh u$ and $\sinh u$ of the n-complex variable u can be defined by the series

$$\cosh u = 1 + u^2/2! + u^4/4! + \dots, \quad (131)$$

$$\sinh u = u + u^3/3! + u^5/5! + \dots. \quad (132)$$

It can be checked by series multiplication that the usual addition theorems hold for the n-complex numbers u, u' ,

$$\cosh(u + u') = \cosh u \cosh u' + \sinh u \sinh u', \quad (133)$$

$$\sinh(u + u') = \sinh u \cosh u' + \cosh u \sinh u'. \quad (134)$$

In order to obtain expressions for the hyperbolic functions of n-complex variables, these will be expressed as

$$\cosh u = \frac{1}{2}(e^u + e^{-u}), \quad \sinh u = \frac{1}{2}(e^u - e^{-u}). \quad (135)$$

The validity of Eq. (135) can be checked by comparing the series for the two sides of the relations. Since the expression of the exponential function $e^{h_k y}$ in terms of the units $1, h_1, \dots, h_{n-1}$ given in Eq. (53) depends on the polar cosexponential functions $g_{np}(y)$, the expression of the hyperbolic functions will depend on the even part $g_{p+}(y) = (1/2)[g_{np}(y) + g_{np}(-y)]$ and on the odd part $g_{p-}(y) = (1/2)[g_{np}(y) - g_{np}(-y)]$ of g_{np} ,

$$\cosh(h_k y) = \sum_{p=0}^{n-1} h_{kp-n[kp/n]} g_{p+}(y), \quad (136)$$

$$\sinh(h_k y) = \sum_{p=0}^{n-1} h_{kp-n[kp/n]} g_{p-}(y), \quad (137)$$

where

$$g_{p+}(y) = \frac{1}{n} \sum_{l=0}^{n-1} \left\{ \cosh \left[y \cos \left(\frac{2\pi l}{n} \right) \right] \cos \left[y \sin \left(\frac{2\pi l}{n} \right) \right] \cos \left(\frac{2\pi l p}{n} \right) + \sinh \left[y \cos \left(\frac{2\pi l}{n} \right) \right] \sin \left[y \sin \left(\frac{2\pi l}{n} \right) \right] \sin \left(\frac{2\pi l p}{n} \right) \right\}, \quad (138)$$

$$g_{p-}(y) = \frac{1}{n} \sum_{l=0}^{n-1} \left\{ \sinh \left[y \cos \left(\frac{2\pi l}{n} \right) \right] \cos \left[y \sin \left(\frac{2\pi l}{n} \right) \right] \cos \left(\frac{2\pi l p}{n} \right) + \cosh \left[y \cos \left(\frac{2\pi l}{n} \right) \right] \sin \left[y \sin \left(\frac{2\pi l}{n} \right) \right] \sin \left(\frac{2\pi l p}{n} \right) \right\}. \quad (139)$$

The exponential, trigonometric and hyperbolic functions can also be expressed with the aid of the bases introduced in Eqs. (82) and (83). Using the expression of the n-complex number in Eq. (87), for even n , yields for the exponential of the n-complex variable u

$$e^u = e_+ e^{v_+} + e_- e^{v_-} + \sum_{k=1}^{n/2-1} e^{v_k} (e_k \cos \tilde{v}_k + \tilde{e}_k \sin \tilde{v}_k). \quad (140)$$

For odd n , the expression of the n-complex variable in Eq. (88) yields for the exponential

$$e^u = e_+ e^{v_+} + \sum_{k=1}^{(n-1)/2} e^{v_k} (e_k \cos \tilde{v}_k + \tilde{e}_k \sin \tilde{v}_k). \quad (141)$$

The trigonometric functions can be obtained from Eqs. (140) and (141) with the aid of Eqs. (126). The trigonometric functions of the n-complex variable u are, for even n ,

$$\cos u = e_+ \cos v_+ + e_- \cos v_- + \sum_{k=1}^{n/2-1} (e_k \cos v_k \cosh \tilde{v}_k - \tilde{e}_k \sin v_k \sinh \tilde{v}_k), \quad (142)$$

$$\sin u = e_+ \sin v_+ + e_- \sin v_- + \sum_{k=1}^{n/2-1} (e_k \sin v_k \cosh \tilde{v}_k + \tilde{e}_k \cos v_k \sinh \tilde{v}_k), \quad (143)$$

and for odd n the trigonometric functions are

$$\cos u = e_+ \cos v_+ + \sum_{k=1}^{(n-1)/2} (e_k \cos v_k \cosh \tilde{v}_k - \tilde{e}_k \sin v_k \sinh \tilde{v}_k), \quad (144)$$

$$\sin u = e_+ \sin v_+ + \sum_{k=1}^{(n-1)/2} (e_k \sin v_k \cosh \tilde{v}_k + \tilde{e}_k \cos v_k \sinh \tilde{v}_k). \quad (145)$$

The hyperbolic functions can be obtained from Eqs. (140) and (141 with the aid of Eqs. (135). The hyperbolic functions of the n -complex variable u are, for even n ,

$$\cosh u = e_+ \cosh v_+ + e_- \cosh v_- + \sum_{k=1}^{n/2-1} (e_k \cosh v_k \cos \tilde{v}_k + \tilde{e}_k \sinh v_k \sin \tilde{v}_k), \quad (146)$$

$$\sinh u = e_+ \sinh v_+ + e_- \sinh v_- + \sum_{k=1}^{n/2-1} (e_k \sinh v_k \cos \tilde{v}_k + \tilde{e}_k \cosh v_k \sin \tilde{v}_k), \quad (147)$$

and for odd n the hyperbolic functions are

$$\cosh u = e_+ \cosh v_+ + \sum_{k=1}^{(n-1)/2} (e_k \cosh v_k \cos \tilde{v}_k + \tilde{e}_k \sinh v_k \sin \tilde{v}_k), \quad (148)$$

$$\sinh u = e_+ \sinh v_+ + \sum_{k=1}^{(n-1)/2} (e_k \sinh v_k \cos \tilde{v}_k + \tilde{e}_k \cosh v_k \sin \tilde{v}_k). \quad (149)$$

7 Power series of polar n -complex numbers

An n -complex series is an infinite sum of the form

$$a_0 + a_1 + a_2 + \cdots + a_n + \cdots, \quad (150)$$

where the coefficients a_n are n -complex numbers. The convergence of the series (150) can be defined in terms of the convergence of its n real components. The convergence of a n -complex series can also be studied using n -complex variables. The main criterion for absolute convergence remains the comparison theorem, but this requires a number of inequalities which will be discussed further.

The modulus $d = |u|$ of an n -complex number u has been defined in Eq. (20). Since $|x_0| \leq |u|, |x_1| \leq |u|, \dots, |x_{n-1}| \leq |u|$, a property of absolute convergence established via a comparison theorem based on the modulus of the series (150) will ensure the absolute convergence of each real component of that series.

The modulus of the sum $u_1 + u_2$ of the n -complex numbers u_1, u_2 fulfils the inequality

$$||u'| - |u''|| \leq |u' + u''| \leq |u'| + |u''|. \quad (151)$$

For the product, the relation is

$$|u'u''| \leq \sqrt{n}|u'||u''|, \quad (152)$$

as can be shown from Eqs. (28) and (29). The relation (152) replaces the relation of equality extant between 2-dimensional regular complex numbers. The equality in Eq. (152) takes place for $\rho_1\rho'_1 = 0, \dots, \rho_{[(n-1)/2]}\rho'_{[(n-1)/2]} = 0$ and, for even n , for $v_+v'_- = 0, v_-v'_+ = 0$.

From Eq. (152) it results, for $u = u'$, that

$$|u^2| \leq \sqrt{n}|u|^2. \quad (153)$$

The relation in Eq. (153) becomes an equality for $\rho_1 = 0, \dots, \rho_{[(n-1)/2]} = 0$ and, for even n , $v_+ = 0$ or $v_- = 0$. The inequality in Eq. (152) implies that

$$|u^l| \leq n^{(l-1)/2}|u|^l, \quad (154)$$

where l is a natural number. From Eqs. (152) and (154) it results that

$$|au^l| \leq n^{l/2}|a||u|^l. \quad (155)$$

A power series of the n -complex variable u is a series of the form

$$a_0 + a_1u + a_2u^2 + \dots + a_lu^l + \dots. \quad (156)$$

Since

$$\left| \sum_{l=0}^{\infty} a_lu^l \right| \leq \sum_{l=0}^{\infty} n^{l/2}|a_l||u|^l, \quad (157)$$

a sufficient condition for the absolute convergence of this series is that

$$\lim_{l \rightarrow \infty} \frac{\sqrt{n}|a_{l+1}||u|}{|a_l|} < 1. \quad (158)$$

Thus the series is absolutely convergent for

$$|u| < c, \quad (159)$$

where

$$c = \lim_{l \rightarrow \infty} \frac{|a_l|}{\sqrt{n}|a_{l+1}|}. \quad (160)$$

The convergence of the series (156) can be also studied with the aid of the formulas (119), (120) which for integer values of m are valid for any values of x_0, \dots, x_{n-1} , as mentioned previously. If $a_l = \sum_{p=0}^{n-1} h_p a_{lp}$, and

$$A_{l+} = \sum_{p=0}^{n-1} a_{lp}, \quad (161)$$

$$A_{lk} = \sum_{p=0}^{n-1} a_{lp} \cos \frac{2\pi kp}{n}, \quad (162)$$

$$\tilde{A}_{lk} = \sum_{p=0}^{n-1} a_{lp} \sin \frac{2\pi kp}{n}, \quad (163)$$

for $k = 1, \dots, [(n-1)/2]$, and for even n

$$A_{l-} = \sum_{p=0}^{n-1} (-1)^p a_{lp}, \quad (164)$$

the series (156) can be written, for even n , as

$$\sum_{l=0}^{\infty} \left[e_+ A_{l+} v_+^l + e_- A_{l-} v_-^l + \sum_{k=1}^{n/2-1} (e_k A_{lk} + \tilde{e}_k \tilde{A}_{lk}) (e_k v_k + \tilde{e}_k \tilde{v}_k)^l \right], \quad (165)$$

and for odd n as

$$\sum_{l=0}^{\infty} \left[e_+ A_{l+} v_+^l + \sum_{k=1}^{(n-1)/2} (e_k A_{lk} + \tilde{e}_k \tilde{A}_{lk}) (e_k v_k + \tilde{e}_k \tilde{v}_k)^l \right]. \quad (166)$$

The series in Eq. (156) is absolutely convergent for

$$|v_+| < c_+, |v_-| < c_-, \rho_k < c_k, \quad (167)$$

for $k = 1, \dots, [(n-1)/2]$, where

$$c_+ = \lim_{l \rightarrow \infty} \frac{|A_{l+}|}{|A_{l+1,+}|}, c_- = \lim_{l \rightarrow \infty} \frac{|A_{l-}|}{|A_{l+1,-}|}, c_k = \lim_{l \rightarrow \infty} \frac{(A_{lk}^2 + \tilde{A}_{lk}^2)^{1/2}}{(A_{l+1,k}^2 + \tilde{A}_{l+1,k}^2)^{1/2}}. \quad (168)$$

The relations (167) show that the region of convergence of the series (156) is an n -dimensional cylinder.

It can be shown that, for even n , $c = (1/\sqrt{n}) \min(c_+, c_-, c_1, \dots, c_{n/2-1})$, and for odd n $c = (1/\sqrt{n}) \min(c_+, c_1, \dots, c_{(n-1)/2})$, where \min designates the smallest of the numbers in the argument of this function. Using the expression of $|u|$ in Eqs. (28) or (29), it can be seen that the spherical region of convergence defined in Eqs. (159), (160) is a subset of the cylindrical region of convergence defined in Eqs. (167) and (168).

8 Analytic functions of polar n-complex variables

The derivative of a function $f(u)$ of the n-complex variables u is defined as a function $f'(u)$ having the property that

$$|f(u) - f(u_0) - f'(u_0)(u - u_0)| \rightarrow 0 \text{ as } |u - u_0| \rightarrow 0. \quad (169)$$

If the difference $u - u_0$ is not parallel to one of the nodal hypersurfaces, the definition in Eq. (169) can also be written as

$$f'(u_0) = \lim_{u \rightarrow u_0} \frac{f(u) - f(u_0)}{u - u_0}. \quad (170)$$

The derivative of the function $f(u) = u^m$, with m an integer, is $f'(u) = mu^{m-1}$, as can be seen by developing $u^m = [u_0 + (u - u_0)]^m$ as

$$u^m = \sum_{p=0}^m \frac{m!}{p!(m-p)!} u_0^{m-p} (u - u_0)^p, \quad (171)$$

and using the definition (169).

If the function $f'(u)$ defined in Eq. (169) is independent of the direction in space along which u is approaching u_0 , the function $f(u)$ is said to be analytic, analogously to the case of functions of regular complex variables. [7] The function u^m , with m an integer, of the n-complex variable u is analytic, because the difference $u^m - u_0^m$ is always proportional to $u - u_0$, as can be seen from Eq. (171). Then series of integer powers of u will also be analytic functions of the n-complex variable u , and this result holds in fact for any commutative algebra.

If an analytic function is defined by a series around a certain point, for example $u = 0$, as

$$f(u) = \sum_{k=0}^{\infty} a_k u^k, \quad (172)$$

an expansion of $f(u)$ around a different point u_0 ,

$$f(u) = \sum_{k=0}^{\infty} c_k (u - u_0)^k, \quad (173)$$

can be obtained by substituting in Eq. (172) the expression of u^k according to Eq. (171). Assuming that the series are absolutely convergent so that the order of the terms can be

modified and ordering the terms in the resulting expression according to the increasing powers of $u - u_0$ yields

$$f(u) = \sum_{k,l=0}^{\infty} \frac{(k+l)!}{k!l!} a_{k+l} u_0^l (u - u_0)^k. \quad (174)$$

Since the derivative of order k at $u = u_0$ of the function $f(u)$, Eq. (172), is

$$f^{(k)}(u_0) = \sum_{l=0}^{\infty} \frac{(k+l)!}{l!} a_{k+l} u_0^l, \quad (175)$$

the expansion of $f(u)$ around $u = u_0$, Eq. (174), becomes

$$f(u) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(u_0) (u - u_0)^k, \quad (176)$$

which has the same form as the series expansion of 2-dimensional complex functions. The relation (176) shows that the coefficients in the series expansion, Eq. (173), are

$$c_k = \frac{1}{k!} f^{(k)}(u_0). \quad (177)$$

The rules for obtaining the derivatives and the integrals of the basic functions can be obtained from the series of definitions and, as long as these series expansions have the same form as the corresponding series for the 2-dimensional complex functions, the rules of derivation and integration remain unchanged.

If the n -complex function $f(u)$ of the n -complex variable u is written in terms of the real functions $P_k(x_0, \dots, x_{n-1})$, $k = 0, 1, \dots, n-1$ of the real variables x_0, x_1, \dots, x_{n-1} as

$$f(u) = \sum_{k=0}^{n-1} h_k P_k(x_0, \dots, x_{n-1}), \quad (178)$$

then relations of equality exist between the partial derivatives of the functions P_k . The derivative of the function f can be written as

$$\lim_{\Delta u \rightarrow 0} \frac{1}{\Delta u} \sum_{k=0}^{n-1} \left(h_k \sum_{l=0}^{n-1} \frac{\partial P_k}{\partial x_l} \Delta x_l \right), \quad (179)$$

where

$$\Delta u = \sum_{k=0}^{n-1} h_l \Delta x_l. \quad (180)$$

The relations between the partials derivatives of the functions P_k are obtained by setting successively in Eq. (179) $\Delta u = h_l \Delta x_l$, for $l = 0, 1, \dots, n-1$, and equating the resulting expressions. The relations are

$$\frac{\partial P_k}{\partial x_0} = \frac{\partial P_{k+1}}{\partial x_1} = \dots = \frac{\partial P_{n-1}}{\partial x_{n-k-1}} = \frac{\partial P_0}{\partial x_{n-k}} = \dots = \frac{\partial P_{k-1}}{\partial x_{n-1}}, \quad (181)$$

for $k = 0, 1, \dots, n-1$. The relations (181) are analogous to the Riemann relations for the real and imaginary components of a complex function. It can be shown from Eqs. (181) that the components P_k fulfil the second-order equations

$$\begin{aligned} \frac{\partial^2 P_k}{\partial x_0 \partial x_l} &= \frac{\partial^2 P_k}{\partial x_1 \partial x_{l-1}} = \dots = \frac{\partial^2 P_k}{\partial x_{[l/2]} \partial x_{l-[l/2]}} \\ &= \frac{\partial^2 P_k}{\partial x_{l+1} \partial x_{n-1}} = \frac{\partial^2 P_k}{\partial x_{l+2} \partial x_{n-2}} = \dots = \frac{\partial^2 P_k}{\partial x_{l+1+[(n-l-2)/2]} \partial x_{n-1-[(n-l-2)/2]}}, \end{aligned} \quad (182)$$

for $k, l = 0, 1, \dots, n-1$.

9 Integrals of polar n-complex functions

The singularities of n-complex functions arise from terms of the form $1/(u-u_0)^n$, with $n > 0$. Functions containing such terms are singular not only at $u = u_0$, but also at all points of the hypersurfaces passing through the pole u_0 and which are parallel to the nodal hypersurfaces.

The integral of an n-complex function between two points A, B along a path situated in a region free of singularities is independent of path, which means that the integral of an analytic function along a loop situated in a region free of singularities is zero,

$$\oint_{\Gamma} f(u) du = 0, \quad (183)$$

where it is supposed that a surface Σ spanning the closed loop Γ is not intersected by any of the hypersurfaces associated with the singularities of the function $f(u)$. Using the expression, Eq. (178), for $f(u)$ and the fact that

$$du = \sum_{k=0}^{n-1} h_k dx_k, \quad (184)$$

the explicit form of the integral in Eq. (183) is

$$\oint_{\Gamma} f(u) du = \oint_{\Gamma} \sum_{k=0}^{n-1} h_k \sum_{l=0}^{n-1} P_l dx_{k-l+n[(n-k-1+l)/n]}. \quad (185)$$

If the functions P_k are regular on a surface Σ spanning the loop Γ , the integral along the loop Γ can be transformed in an integral over the surface Σ of terms of the form $\partial P_l / \partial x_{k-m+n[(n-k+m-1)/n]} - \partial P_m / \partial x_{k-l+n[(n-k+l-1)/n]}$. These terms are equal to zero by Eqs. (181), and this proves Eq. (183).

The integral of the function $(u - u_0)^m$ on a closed loop Γ is equal to zero for m a positive or negative integer not equal to -1,

$$\oint_{\Gamma} (u - u_0)^m du = 0, \quad m \text{ integer}, \quad m \neq -1. \quad (186)$$

This is due to the fact that $\int (u - u_0)^m du = (u - u_0)^{m+1}/(m+1)$, and to the fact that the function $(u - u_0)^{m+1}$ is singlevalued for m an integer.

The integral $\oint_{\Gamma} du/(u - u_0)$ can be calculated using the exponential form, Eqs. (99) and (101), for the difference $u - u_0$, which for even n is

$$u - u_0 = \rho \exp \left\{ \sum_{p=1}^{n-1} h_p \left[\frac{1}{n} \ln \frac{\sqrt{2}}{\tan \theta_+} + \frac{(-1)^p}{n} \ln \frac{\sqrt{2}}{\tan \theta_-} - \frac{2}{n} \sum_{k=2}^{n/2-1} \cos \left(\frac{2\pi kp}{n} \right) \ln \tan \psi_{k-1} \right] + \sum_{k=1}^{n/2-1} \tilde{e}_k \phi_k \right\}, \quad (187)$$

and for odd n is

$$u - u_0 = \rho \exp \left\{ \sum_{p=1}^{n-1} h_p \left[\frac{1}{n} \ln \frac{\sqrt{2}}{\tan \theta_+} - \frac{2}{n} \sum_{k=2}^{(n-1)/2} \cos \left(\frac{2\pi kp}{n} \right) \ln \tan \psi_{k-1} \right] + \sum_{k=1}^{(n-1)/2} \tilde{e}_k \phi_k \right\}. \quad (188)$$

Thus for even n the quantity $du/(u - u_0)$ is

$$\frac{du}{u - u_0} = \frac{d\rho}{\rho} + \sum_{p=1}^{n-1} h_p \left[\frac{1}{n} d \ln \frac{\sqrt{2}}{\tan \theta_+} + \frac{(-1)^p}{n} d \ln \frac{\sqrt{2}}{\tan \theta_-} - \frac{2}{n} \sum_{k=2}^{n/2-1} \cos \left(\frac{2\pi kp}{n} \right) d \ln \tan \psi_{k-1} \right] + \sum_{k=1}^{n/2-1} \tilde{e}_k d\phi_k, \quad (189)$$

and for odd n

$$\frac{du}{u - u_0} = \frac{d\rho}{\rho} + \sum_{p=1}^{n-1} h_p \left[\frac{1}{n} d \ln \frac{\sqrt{2}}{\tan \theta_+} - \frac{2}{n} \sum_{k=2}^{(n-1)/2} \cos \left(\frac{2\pi kp}{n} \right) d \ln \tan \psi_{k-1} \right] + \sum_{k=1}^{(n-1)/2} \tilde{e}_k d\phi_k. \quad (190)$$

Since $\rho, \ln(\sqrt{2}/\tan \theta_+), \ln(\sqrt{2}/\tan \theta_-), \ln(\tan \psi_{k-1})$ are singlevalued variables, it follows that $\oint_{\Gamma} d\rho/\rho = 0, \oint_{\Gamma} d(\ln \sqrt{2}/\tan \theta_+) = 0, \oint_{\Gamma} d(\ln \sqrt{2}/\tan \theta_-) = 0, \oint_{\Gamma} d(\ln \tan \psi_{k-1}) = 0$. On the other hand since, ϕ_k are cyclic variables, they may give contributions to the integral around the closed loop Γ .

The expression of $\oint_{\Gamma} du/(u - u_0)$ can be written with the aid of a functional which will be called $\text{int}(M, C)$, defined for a point M and a closed curve C in a two-dimensional plane, such that

$$\text{int}(M, C) = \begin{cases} 1 & \text{if } M \text{ is an interior point of } C, \\ 0 & \text{if } M \text{ is exterior to } C. \end{cases} \quad (191)$$

With this notation the result of the integration on a closed path Γ can be written as

$$\oint_{\Gamma} \frac{du}{u - u_0} = \sum_{k=1}^{[(n-1)/2]} 2\pi \tilde{e}_k \text{int}(u_{0\xi_k\eta_k}, \Gamma_{\xi_k\eta_k}), \quad (192)$$

where $u_{0\xi_k\eta_k}$ and $\Gamma_{\xi_k\eta_k}$ are respectively the projections of the point u_0 and of the loop Γ on the plane defined by the axes ξ_k and η_k , as shown in Fig. 3.

If $f(u)$ is an analytic n -complex function which can be expanded in a series as written in Eq. (173), and the expansion holds on the curve Γ and on a surface spanning Γ , then from Eqs. (186) and (192) it follows that

$$\oint_{\Gamma} \frac{f(u)du}{u - u_0} = 2\pi f(u_0) \sum_{k=1}^{[(n-1)/2]} \tilde{e}_k \text{int}(u_{0\xi_k\eta_k}, \Gamma_{\xi_k\eta_k}). \quad (193)$$

Substituting in the right-hand side of Eq. (193) the expression of $f(u)$ in terms of the real components P_k , Eq. (178), yields

$$\oint_{\Gamma} \frac{f(u)du}{u - u_0} = \frac{2}{n} \sum_{k=1}^{[(n-1)/2]} \sum_{l,m=0}^{n-1} h_l \sin \left[\frac{2\pi(l-m)k}{n} \right] P_m(u_0) \text{int}(u_{0\xi_k\eta_k}, \Gamma_{\xi_k\eta_k}). \quad (194)$$

It the integral in Eq. (194) is written as

$$\oint_{\Gamma} \frac{f(u)du}{u - u_0} = \sum_{l=0}^{n-1} h_l I_l, \quad (195)$$

it can be checked that

$$\sum_{l=0}^{n-1} I_l = 0. \quad (196)$$

If $f(u)$ can be expanded as written in Eq. (173) on Γ and on a surface spanning Γ , then from Eqs. (186) and (192) it also results that

$$\oint_{\Gamma} \frac{f(u)du}{(u - u_0)^{n+1}} = \frac{2\pi}{n!} f^{(n)}(u_0) \sum_{k=1}^{[(n-1)/2]} \tilde{e}_k \text{int}(u_{0\xi_k\eta_k}, \Gamma_{\xi_k\eta_k}), \quad (197)$$

where the fact has been used that the derivative $f^{(n)}(u_0)$ is related to the expansion coefficient in Eq. (173) according to Eq. (177).

If a function $f(u)$ is expanded in positive and negative powers of $u - u_l$, where u_l are n-complex constants, l being an index, the integral of f on a closed loop Γ is determined by the terms in the expansion of f which are of the form $r_l/(u - u_l)$,

$$f(u) = \cdots + \sum_l \frac{r_l}{u - u_l} + \cdots. \quad (198)$$

Then the integral of f on a closed loop Γ is

$$\oint_{\Gamma} f(u) du = 2\pi \sum_l \sum_{k=1}^{[(n-1)/2]} \tilde{e}_k \text{int}(u_{l\xi_k\eta_k}, \Gamma_{\xi_k\eta_k}) r_l. \quad (199)$$

10 Factorization of polar n-complex polynomials

A polynomial of degree m of the n-complex variable u has the form

$$P_m(u) = u^m + a_1 u^{m-1} + \cdots + a_{m-1} u + a_m, \quad (200)$$

where a_l , for $l = 1, \dots, m$, are in general n-complex constants. If $a_l = \sum_{p=0}^{n-1} h_p a_{lp}$, and with the notations of Eqs. (161)-(164) applied for $l = 1, \dots, m$, the polynomial $P_m(u)$ can be written, for even n , as

$$\begin{aligned} P_m = & e_+ \left(v_+^m + \sum_{l=1}^m A_{l+} v_+^{m-l} \right) + e_- \left(v_-^m + \sum_{l=1}^m A_{l-} v_-^{m-l} \right) \\ & + \sum_{k=1}^{n/2-1} \left[(e_k v_k + \tilde{e}_k \tilde{v}_k)^m + \sum_{l=1}^m (e_k A_{lk} + \tilde{e}_k \tilde{A}_{lk}) (e_k v_k + \tilde{e}_k \tilde{v}_k)^{m-l} \right], \end{aligned} \quad (201)$$

where the constants $A_{l+}, A_{l-}, A_{lk}, \tilde{A}_{lk}$ are real numbers. For odd n the expression of the polynomial is

$$\begin{aligned} P_m = & e_+ \left(v_+^m + \sum_{l=1}^m A_{l+} v_+^{m-l} \right) \\ & + \sum_{k=1}^{(n-1)/2} \left[(e_k v_k + \tilde{e}_k \tilde{v}_k)^m + \sum_{l=1}^m (e_k A_{lk} + \tilde{e}_k \tilde{A}_{lk}) (e_k v_k + \tilde{e}_k \tilde{v}_k)^{m-l} \right]. \end{aligned} \quad (202)$$

The polynomials of degree m in $e_k v_k + \tilde{e}_k \tilde{v}_k$ in Eqs. (201) and (202) can always be written as a product of linear factors of the form $e_k(v_k - v_{kp}) + \tilde{e}_k(\tilde{v}_k - \tilde{v}_{kp})$, where the constants v_{kp}, \tilde{v}_{kp} are real. The polynomials of degree m with real coefficients in Eqs. (201) and (202) which are multiplied by e_+ and e_- can be written as a product of linear or quadratic factors with real coefficients, or as a product of linear factors which, if imaginary, appear always in

complex conjugate pairs. Using the latter form for the simplicity of notations, the polynomial P_m can be written, for even n , as

$$P_m = e_+ \prod_{p=1}^m (v_+ - v_{p+}) + e_- \prod_{p=1}^m (v_- - v_{p-}) + \sum_{k=1}^{n/2-1} \prod_{p=1}^m \{e_k(v_k - v_{kp}) + \tilde{e}_k(\tilde{v}_k - \tilde{v}_{kp})\}, \quad (203)$$

where the quantities v_{p+} appear always in complex conjugate pairs, and the quantities \tilde{v}_{p-} appear always in complex conjugate pairs. For odd n the polynomial can be written as

$$P_m = e_+ \prod_{p=1}^m (v_+ - v_{p+}) + \sum_{k=1}^{(n-1)/2} \prod_{p=1}^m \{e_k(v_k - v_{kp}) + \tilde{e}_k(\tilde{v}_k - \tilde{v}_{kp})\}, \quad (204)$$

where the quantities v_{p+} appear always in complex conjugate pairs. Due to the relations (84),(85), the polynomial $P_m(u)$ can be written, for even n , as a product of factors of the form

$$P_m(u) = \prod_{p=1}^m \left\{ e_+(v_+ - v_{p+}) + e_-(v_- - v_{p-}) + \sum_{k=1}^{n/2-1} \{e_k(v_k - v_{kp}) + \tilde{e}_k(\tilde{v}_k - \tilde{v}_{kp})\} \right\}. \quad (205)$$

For odd n , the polynomial $P_m(u)$ can be written as the product

$$P_m(u) = \prod_{p=1}^m \left\{ e_+(v_+ - v_{p+}) + \sum_{k=1}^{(n-1)/2} \{e_k(v_k - v_{kp}) + \tilde{e}_k(\tilde{v}_k - \tilde{v}_{kp})\} \right\}. \quad (206)$$

These relations can be written with the aid of Eqs. (87) and (88) as

$$P_m(u) = \prod_{p=1}^m (u - u_p), \quad (207)$$

where, for even n ,

$$u_p = e_+ v_{p+} + e_- v_{p-} + \sum_{k=1}^{n/2-1} (e_k v_{kp} + \tilde{e}_k \tilde{v}_{kp}), \quad (208)$$

and for odd n

$$u_p = e_+ v_{p+} + \sum_{k=1}^{(n-1)/2} (e_k v_{kp} + \tilde{e}_k \tilde{v}_{kp}), \quad (209)$$

for $p = 1, \dots, m$. The roots v_{p+} , the roots v_{p-} and, for a given k , the roots $e_k v_{k1} + \tilde{e}_k \tilde{v}_{k1}, \dots, e_k v_{km} + \tilde{e}_k \tilde{v}_{km}$ defined in Eqs. (203) or (204) may be ordered arbitrarily. This means that Eqs. (208) or (209) give sets of m roots u_1, \dots, u_m of the polynomial $P_m(u)$, corresponding to the various ways in which the roots $v_{p+}, v_{p-}, e_k v_{kp} + \tilde{e}_k \tilde{v}_{kp}$ are ordered

according to p in each group. Thus, while the n -complex components in Eq. (202) taken separately have unique factorizations, the polynomial $P_m(u)$ can be written in many different ways as a product of linear factors.

If $P(u) = u^2 - 1$, the degree is $m = 2$, the coefficients of the polynomial are $a_1 = 0, a_2 = -1$, the n -complex components of a_2 are $a_{20} = -1, a_{21} = 0, \dots, a_{2,n-1} = 0$, the components $A_{2+}, A_{2-}, A_{2k}, \tilde{A}_{2k}$ calculated according to Eqs. (161)-(164) are $A_{2+} = -1, A_{2-} = -1, A_{2k} = -1, \tilde{A}_{2k} = 0, k = 1, \dots, [(n-1)/2]$. The expression of $P(u)$ for even n , Eq. (201), is $e_+(v_+^2 - 1) + e_-(v_-^2 - 1) + \sum_{k=1}^{n/2-1} \{(e_k v_k + \tilde{e}_k \tilde{v}_k)^2 - e_k\}$, and Eq. (203) has the form $u^2 - 1 = e_+(v_+ + 1)(v_+ - 1) + e_-(v_- + 1)(v_- - 1) + \sum_{k=1}^{n/2-1} \{e_k(v_k + 1) + \tilde{e}_k \tilde{v}_k\} \{e_k(v_k - 1) + \tilde{e}_k \tilde{v}_k\}$. For odd n , the expression of $P(u)$, Eq. (202), is $e_+(v_+^2 - 1) + \sum_{k=1}^{(n-1)/2} \{(e_k v_k + \tilde{e}_k \tilde{v}_k)^2 - e_k\}$, and Eq. (204) has the form $u^2 - 1 = e_+(v_+ + 1)(v_+ - 1) + \sum_{k=1}^{(n-1)/2} \{e_k(v_k + 1) + \tilde{e}_k \tilde{v}_k\} \{e_k(v_k - 1) + \tilde{e}_k \tilde{v}_k\}$. The factorization in Eq. (207) is $u^2 - 1 = (u - u_1)(u - u_2)$, where for even n , $u_1 = \pm e_+ \pm e_- \pm e_1 \pm e_2 \pm \dots \pm e_{n/2-1}, u_2 = -u_1$, so that there are $2^{n/2}$ independent sets of roots u_1, u_2 of $u^2 - 1$. It can be checked that $(\pm e_+ \pm e_- \pm e_1 \pm e_2 \pm \dots \pm e_{n/2-1})^2 = e_+ + e_- + e_1 + e_2 + \dots + e_{n/2-1} = 1$. For odd n , $u_1 = \pm e_+ \pm e_1 \pm e_2 \pm \dots \pm e_{(n-1)/2}, u_2 = -u_1$, so that there are $2^{(n-1)/2}$ independent sets of roots u_1, u_2 of $u^2 - 1$. It can be checked that $(\pm e_+ \pm e_1 \pm e_2 \pm \dots \pm e_{(n-1)/2})^2 = e_+ + e_1 + e_2 + \dots + e_{(n-1)/2} = 1$.

11 Representation of polar n -complex numbers by irreducible matrices

If the unitary matrix written in Eq. (21), for even n , is called T_e , and the unitary matrix written in Eq. (22), for odd n , is called T_o , it can be shown that, for even n , the matrix $T_e U T_e^{-1}$ has the form

$$T_e U T_e^{-1} = \begin{pmatrix} v_+ & 0 & 0 & \cdots & 0 \\ 0 & v_- & 0 & \cdots & 0 \\ 0 & 0 & V_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & V_{n/2-1} \end{pmatrix} \quad (210)$$

and, for odd n , the matrix $T_oUT_o^{-1}$ has the form

$$T_oUT_o^{-1} = \begin{pmatrix} v_+ & 0 & 0 & \cdots & 0 \\ 0 & V_1 & 0 & \cdots & 0 \\ 0 & 0 & V_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & V_{(n-1)/2} \end{pmatrix}, \quad (211)$$

where U is the matrix in Eq. (47) used to represent the n -complex number u . In Eqs. (210) and (211), V_k are the matrices

$$V_k = \begin{pmatrix} v_k & \tilde{v}_k \\ -\tilde{v}_k & v_k \end{pmatrix}, \quad (212)$$

for $k = 1, \dots, [(n-1)/2]$, where v_k, \tilde{v}_k are the variables introduced in Eqs. (14) and (15), and the symbols 0 denote, according to the case, the real number zero, or one of the matrices

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (213)$$

The relations between the variables v_k, \tilde{v}_k for the multiplication of n -complex numbers have been written in Eq. (44). The matrices $T_eUT_e^{-1}$ and $T_oUT_o^{-1}$ provide an irreducible representation [8] of the n -complex numbers u in terms of matrices with real coefficients.

12 Conclusions

The operations of addition and multiplication of the n -complex numbers introduced in this work have a geometric interpretation based on the amplitude ρ , the modulus d and the polar, planar and azimuthal angles $\theta_+, \theta_-, \psi_k, \phi_k$. If $x_0 + x_1 + \cdots + x_{n-1} > 0$ and $x_0 - x_1 + \cdots + x_{n-2} - x_{n-1} > 0$, the n -complex numbers can be written in exponential and trigonometric forms with the aid of the modulus, amplitude and the angular variables. The n -complex functions defined by series of powers are analytic, and the partial derivatives of the components of the n -complex functions are closely related. The integrals of n -complex functions are independent of path in regions where the functions are regular. The fact that the exponential form of the n -complex numbers depends on the cyclic variables ϕ_k leads to

the concept of pole and residue for integrals on closed paths. The polynomials of n -complex variables can be written as products of linear or quadratic factors.

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FIGURE CAPTIONS

Fig. 1. Representation of the hypercomplex bases $1, h_1, \dots, h_{n-1}$ by points on a circle at the angles $\alpha_k = 2\pi k/n$. The product $h_j h_k$ will be represented by the point of the circle at the angle $2\pi(j+k)/n$, $i, k = 0, 1, \dots, n-1$. If $2\pi \leq 2\pi(j+k)/n \leq 4\pi$, the point represents the basis h_l of angle $\alpha_l = 2\pi(j+k)/n - 2\pi$.

Fig. 2. Radial distance ρ_k and azimuthal angle ϕ_k in the plane of the axes v_k, \tilde{v}_k , and planar angle ψ_{k-1} between the line OA_{1k} and the 2-dimensional plane defined by the axes v_k, \tilde{v}_k . A_k is the projection of the point A on the plane of the axes v_k, \tilde{v}_k , and A_{1k} is the projection of the point A on the 4-dimensional space defined by the axes $v_1, \tilde{v}_1, v_k, \tilde{v}_k$. The polar angle θ_+ is the angle between the line OA_{1+} and the axis v_+ , where A_{1+} is the projection of the point A on the 3-dimensional space generated by the axes v_1, \tilde{v}_1, v_+ . In an even number of dimensions n there is also a polar angle θ_- , which is the angle between the line OA_{1-} and the axis v_- , where A_{1-} is the projection of the point A on the 3-dimensional space generated by the axes v_1, \tilde{v}_1, v_- .

Fig. 3. Integration path Γ and pole u_0 , and their projections $\Gamma_{\xi_k \eta_k}$ and $u_{0\xi_k \eta_k}$ on the plane $\xi_k \eta_k$.

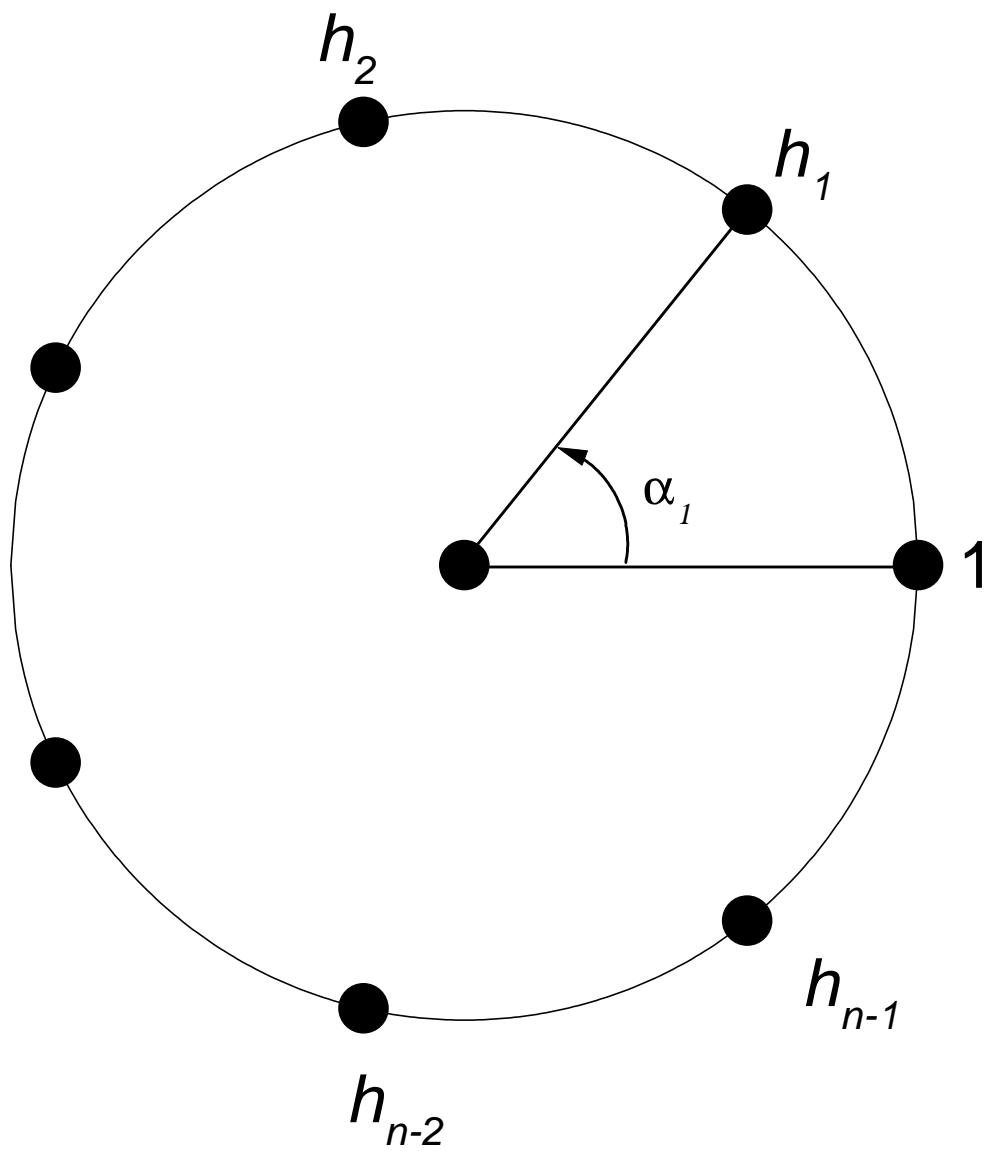


Fig. 1

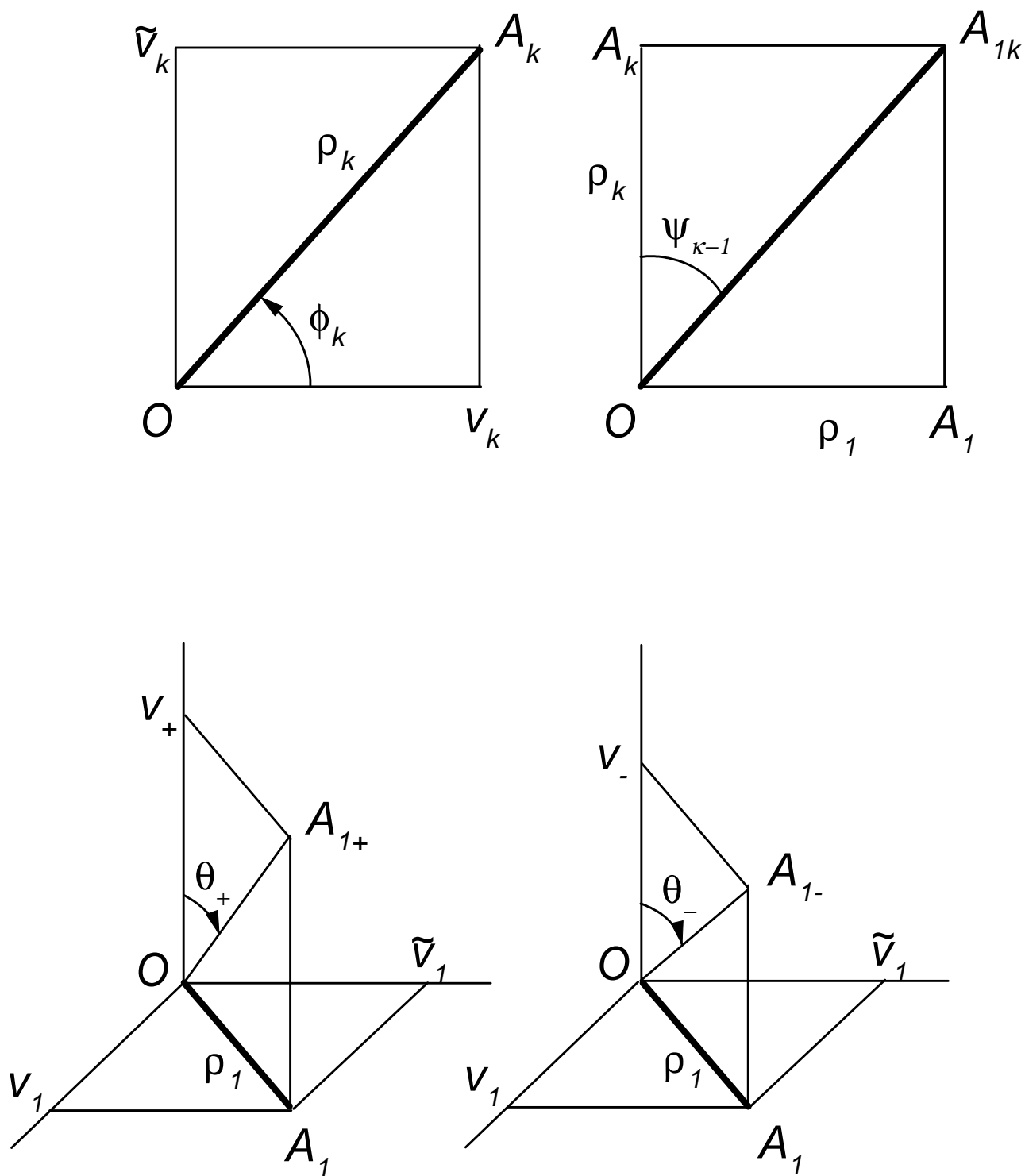


Fig. 2

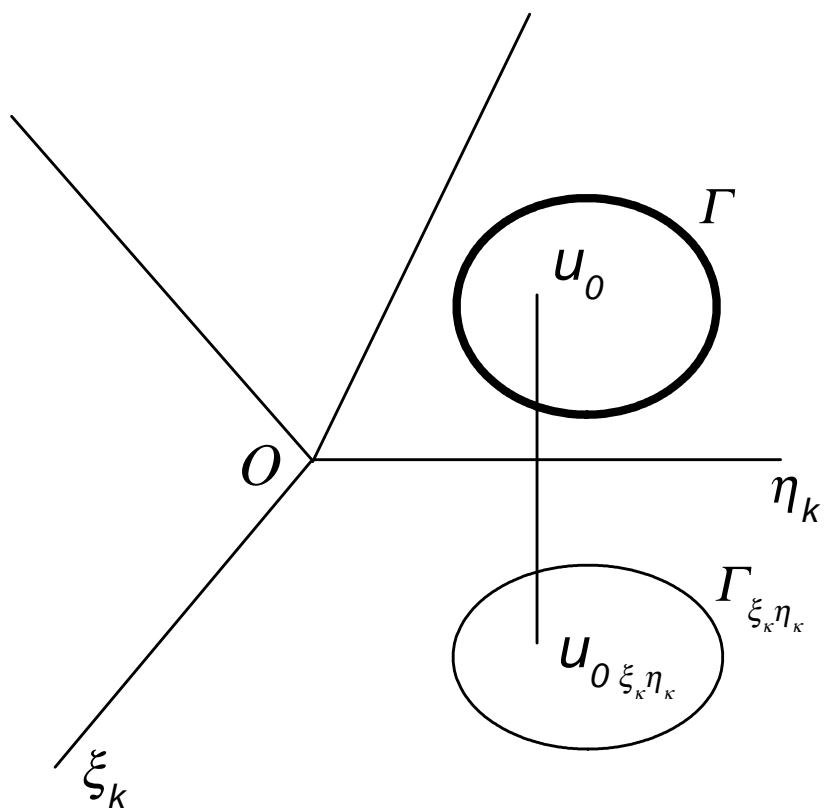


Fig. 3